Optimal control and Convex optimization **Spring 2020**

Lecture 2: Convex reformulation in the Frequency Domain

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They were developed when the author was a postdoc in Prof. Na Li's group at Harvard. Any typos should be sent to* `zhengy@eng.ucsd.edu`.

Learning goals:

1. LQR as a special case of $\mathcal{H}_2$ optimal control;
2. Convex characterization of stabilizing controllers;
3. Transfer matrix characterization of internal stability;
4. System-level synthesis, Input-output parameterization, and Youla;
5. Robust stability;

1 Recap

The problem setup is as follows: we consider continuous-time linear time-invariant (LTI) systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^d, y \in \mathbb{R}^p, z \in \mathbb{R}^q$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively. Consider a dynamic output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ has a state-space realization

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y, \\ u &= C_k \xi + D_k y, \end{aligned} \tag{2}$$

where $\xi \in \mathbb{R}^{n_k}$ is the internal state of controller $\mathbf{K}$.

We have introduced the following optimal control problem

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}, \end{aligned} \tag{3}$$

and its corresponding state-space version is

$$\begin{aligned} \min_{A_k, B_k, C_k, D_k} \quad & \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\| \\ \text{subject to} \quad & \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.} \end{aligned} \tag{4}$$

Both (3) and (4) are non-convex in its present form. Note that the formulation (3) or (4) is very general, including LQR/LQG/$\mathcal{H}_2$/$\mathcal{H}_\infty$ optimal control [8].

In this lecture, we aim to present convex reformulation of (3) by introducing a suitable change of variables. We will characterize the set of stabilizing controllers $\mathcal{C}_{\text{stab}}$, and then look at the cost function.

2 LQR as a special case of $\mathcal{H}_2$ optimal control

The classical Linear Quadratic Regulator has different forms. One typical deterministic form is as follows

$$\begin{aligned} \min \quad & \int_0^\infty x^\mathsf{T} Q x + u^\mathsf{T} R u \; dt \\ \text{subject to} \quad & \dot{x} = Ax + Bu \\ & x(0) = x_0, \end{aligned} \tag{5}$$

where $Q \succ 0, R \succ 0$ are weight matrices and $x_0 \in \mathbb{R}^n$ is the initial value. Another typical stochastic version is

$$\begin{aligned} \min \quad & \mathbb{E}\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T x^\mathsf{T} Q x + u^\mathsf{T} R u \; dt\right] \\ \text{subject to} \quad & \dot{x} = Ax + Bu + w \end{aligned} \tag{6}$$

where $Q \succ 0, R \succ 0$ are weight matrices and $w \sim N(0, I)$ is a Guassian noise. Both (5) and (6) can be reformulated as a special case of $\mathcal{H}_2$ optimal control in the form of (3) or (4).

$\mathcal{H}_2$ norm of transfer matrices:

Given a stable transfer matrix $\mathbf{T} = C(sI - A)^{-1}B$, its $\mathcal{H}_2$ norm is defined as

$$\begin{aligned} \|\mathbf{T}\|_{\mathcal{H}_2}^2 := & \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left(T^*(j\omega)T(j\omega)\right) d\omega \\ = & \int_0^\infty \text{Trace}\left((Ce^{At}B)^\mathsf{T}(Ce^{At}B)\right) dt \end{aligned}$$

where the second equality comes from the Parseval theorem. Although $\|\mathbf{T}\|_{\mathcal{H}_2}$ can, in principle, be computed from its definition above, we have simple state-space characterizations

$$\begin{aligned} \|\mathbf{T}\|_{\mathcal{H}_2}^2 &= \text{Trace}(B^\mathsf{T} Q B), \qquad \text{where } A^\mathsf{T} Q + QA + C^\mathsf{T} C = 0 \\ \|\mathbf{T}\|_{\mathcal{H}_2}^2 &= \text{Trace}(C P_0 C^\mathsf{T}), \qquad \text{where } AP + PA^\mathsf{T} + BB^\mathsf{T} = 0. \end{aligned}$$

Note that Q and P are observability and controllability Gramians. $\mathcal{H}_2$ norm can also be characterized by LMIs, which will be introduced in later lectures. We have two interpretations:

- **Deterministic interpretation:** Let e_k be the standard unit vector and denote the output

$$\dot{x} = Ax, \quad z = Cx, \quad x(0) = Be_k,$$

  by $z_k(t)$. Note that this is the response to an impulse input to the channel k. Since $z_k(t) = Ce^{At}Be_k$, we have

$$\int_0^\infty z_k(t)^\mathsf{T} z_k(t) dt = e_k^\mathsf{T}\left(\int_0^\infty B^\mathsf{T} e^{A^\mathsf{T} t} C^\mathsf{T} C e^{At} B dt\right) e_k.$$

Therefore, squared $\mathcal{H}_2$ norm is energy sum of transients of output responses:

$$\sum_{k=1}^{m} \int_0^\infty z_k(t)^\mathsf{T} z_k(t) dt = \int_0^\infty \text{Trace}\left((Ce^{At}B)^\mathsf{T}(Ce^{At}B)\right) dt = \|\mathbf{T}\|_{\mathcal{H}_2}^2.$$

- **Stochastic interpretation:** If w is white noise and $\dot{x} = Ax + Bw, z = Cx$ then

$$\lim_{t\to\infty} \mathbb{E}\left(z(t)^\mathsf{T} z(t)\right) = \|\mathbf{T}\|_{\mathcal{H}_2}^2$$

The squared $\mathcal{H}_2$-norm equals the asymptotic variance of output.

According to the deterministic and stochastic interpretations of $\mathcal{H}_2$ norm, it it not difficult to show that both (5) and (6) are equivalent to the following problem

$$\begin{aligned} \min_K \quad & \|\mathbf{T}_{zw}\|_{\mathcal{H}_2}^2 \\ \text{subject to} \quad & \dot{x} = Ax + B_1 w + B_2 u \\ & z = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} u \\ & u = Kx, \end{aligned} \tag{7}$$

where $B_1 = I, B_2 = B$. Problem (7) is a special case of (3).

3 External transfer matrix characterization of internal stability

3.1 Static state feedback

Before introducing the dynamic case, we consider the simplified static state case of (1) and (2) as follows

$$\begin{aligned} \dot{x} &= A + B_2 u \\ u &= Kx. \end{aligned}$$

Then the set of stabilizing static state feedback gains are defined as follows

$$\mathcal{C}_{\text{ss}} = \{K \in \mathbb{R}^{m\times n} \mid A + B_2 K \text{ is stable}\}.$$

It is well-known $\mathcal{C}_{\text{ss}}$ is non-convex, but it admits a convex characterization using a change of variable. In particular

$$\begin{aligned} A + B_2 K \text{ is stable} \quad & \Longleftrightarrow \quad \exists P \succ 0,\ (A + B_2 K)^\mathsf{T} P + P(A + B_2 K) \prec 0 \\ & \Longleftrightarrow \quad \exists X \succ 0,\ X(A + B_2 K)^\mathsf{T} + (A + B_2 K)X \prec 0 \\ & \Longleftrightarrow \quad \exists X \succ 0, Y \in \mathbb{R}^{m\times n},\ XA^\mathsf{T} + YB_2^\mathsf{T} + AX + B_2 Y \prec 0 \end{aligned}$$

Therefore, we have

$$\mathcal{C}_{\text{ss}} = \{K = YX^{-1} \mid X \succ 0, Y \in \mathbb{R}^{m\times n},\ XA^\mathsf{T} + YB_2^\mathsf{T} + AX + B_2 Y \prec 0\},$$

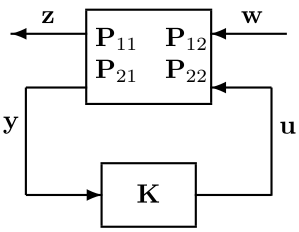

Figure 1: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

where the constraint is convex in terms of the new Lyapunov variables X and Y. We note that the mapping from K to X and Y is not unique in the derivation above, but we can the Lyapunov equality to make the mapping become one-to-one correspondence.

3.2 Dynamic output-feedback

Throughout this document, we denote $\mathcal{RH}_\infty$ as the set all stable real-rational proper transfer matrices, *i.e.*, all poles are on the left open-half complex plane. We have the following standard result [8, Chapter 3].

Lemma 1. *Given a transfer matrix* $\mathbf{T}(s) = C(sI - A)^{-1}B + D$*, we have*

- *If* (A, B, C) *is detectable and stabilizable, then* $\mathbf{T}(s) \in \mathcal{RH}_\infty$ *if and only if* A *is stable;*
- *If* (A, B, C) *is not detectable or stabilizable, then the stability of* A *is sufficient but not necessary for* $\mathbf{T}(s) \in \mathcal{RH}_\infty$.

We have already a state-space characterization:

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable} \right\}, \tag{8}$$

where $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. Unfortunately, the stability condition on A_{cl} in (8) is still non-convex in terms of the parameters (A_k, B_k, C_k, D_k).

There are a few frequency domain characterizations of internal stability. To be precise, let us consider the plant $\mathbf{P}_{22} = C_2(sI - A)^{-1}B_2$,

$$\begin{aligned} \dot{x} &= Ax + B_2 u + \delta_x, \\ y &= C_2 x + \delta_y \end{aligned} \tag{9}$$

and a dynamic controller $\mathbf{u} = \mathbf{K}\mathbf{y} + \delta_u$ with a state-space realization as

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y \\ u &= C_k \xi + D_k y + \delta_u. \end{aligned} \tag{10}$$

It is not difficult to derive the closed-loop responses from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}, \tag{11}$$

where

$$\mathbf{Y} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}, \quad \mathbf{W} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{22}, \quad \mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}, \quad \mathbf{Z} = (I - \mathbf{K}\mathbf{P}_{22})^{-1}.$$

We have a classical transfer matrix characterization of internal stability [8, Lemma 5.3].

Lemma 2. *The system in Figure 1 is internally stable if and only if the transfer matrix from* (δ_y, δ_u) *to* $(\mathbf{y}, \mathbf{u})$ *is stable.*

Proof. Here is a sketch proof for the case of strictly proper plants. It is not difficult the derive a state-space realization of the transfer matrix from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ as

$$\left(\begin{bmatrix}\delta_y \\ \delta_u\end{bmatrix} \to \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix}\right) = \hat{C}_2(zI - \hat{A})^{-1}\hat{B}_2 + \begin{bmatrix} I & 0 \\ D_k & I\end{bmatrix},$$

where

$$\hat{A} = \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}, \; \hat{B}_2 = \begin{bmatrix} B_2 D_k & B_2 \\ B_k & 0\end{bmatrix}, \; \hat{C}_2 = \begin{bmatrix} C_2 & 0 \\ D_k C_2 & C_k\end{bmatrix}.$$

It remains to prove that $(\hat{A}, \hat{B}_2)$ is stabilizable and $(\hat{A}, \hat{C}_2)$ is detectable. Then, the stability of the transfer matrix $\left(\begin{bmatrix}\delta_y \\ \delta_u\end{bmatrix} \to \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix}\right)$ is equivalent to the stability of $\hat{A}$. This completes the proof. □

The stabilizability of $(\hat{A}, \hat{B}_2)$ can be seen from the following fact

$$\begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} + \begin{bmatrix} B_2 D_k & B_2 \\ B_k & 0\end{bmatrix}\begin{bmatrix} -C_2 & F_k \\ F & 0\end{bmatrix} = \begin{bmatrix} A + B_2 F & B_2 C_k + B_2 D_k F_k \\ 0 & A_k + B_k F_k, \end{bmatrix}$$

which will be stable if $A + B_2 F$ and $A_k + B_k F_k$ are stable. The detectability of $(\hat{A}, \hat{C}_2)$ can be shown in a similar way.

We can also look at the closed-loop response from (δ_x, δ_y) to $(\mathbf{x}, \mathbf{u})$. It is not difficult to derive that

$$\begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L}\end{bmatrix}\begin{bmatrix}\delta_x \\ \delta_y\end{bmatrix}, \tag{12}$$

where

$$\mathbf{R} = (zI - A - B_2\mathbf{K}C_2)^{-1}, \quad \mathbf{M} = \mathbf{K}C_2\mathbf{R}, \quad \mathbf{U} = \mathbf{R}B_2\mathbf{K}, \quad \mathbf{L} = \mathbf{K}C_2\mathbf{R}B_2\mathbf{K} + \mathbf{K}.$$

We have a new transfer matrix characterization of internal stability [4].

Lemma 3. *The system in Figure 1 is internally stable if and only if the transfer matrix from* (δ_x, δ_y) *to* $(\mathbf{x}, \mathbf{u})$ *is stable.*

Proof. Let us routinely derive a state-space realization of the transfer matrix from (δ_x, δ_y) to $(\mathbf{x}, \mathbf{u})$ as

$$\left(\begin{bmatrix}\delta_x \\ \delta_y\end{bmatrix} \to \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}\right) = \hat{C}_1(zI - \hat{A})^{-1}\hat{B}_1 + \begin{bmatrix} 0 & 0 \\ 0 & D_k\end{bmatrix},$$

where

$$\hat{B}_1 = \begin{bmatrix} I & B_2 D_k \\ 0 & B_k\end{bmatrix}, \quad \hat{C}_1 = \begin{bmatrix} I & 0 \\ D_k C_2 & C_k\end{bmatrix}$$

It is not difficult to check that $(\hat{A}, \hat{B}_1)$ is stabilizable and $(\hat{A}, \hat{C}_1)$ is detectable. □

Are there other transfer matrix characterizations for internal stability?

The answer is yes; see a recent report [6].

3.3 Two special cases

Here, we show that the transfer matrix characterization of internal stability can be simplified for special cases: 1) open-loop stable plants; 2) the state feedback case. The following result is classical, which is the same as Corollary 5.5 in [8]. For completeness, we provide a proof from a state-space perspective.

Corollary 1. *Consider the system in Figure 1. If the LTI system is open-loop stable (i.e., A is stable), then* $\mathbf{K} \in \mathcal{C}_{stab}$ *if and only if* $(\delta_y \to \mathbf{u}) := \mathbf{U} \in \mathcal{RH}_\infty$.

Proof. The "only if" direction is true by definition. We now prove the sufficiency. We can derive the following state-space representation

$$\mathbf{U} = \begin{bmatrix} D_k C & C_k \end{bmatrix} (zI - \hat{A})^{-1} \begin{bmatrix} BD_k \\ B_k \end{bmatrix} + D_k.$$

Considering the fact that the following matrix

$$\hat{A} + \begin{bmatrix} BD_k \\ B_k \end{bmatrix} \begin{bmatrix} -C & F_k \end{bmatrix} = \begin{bmatrix} A & BC_k + BD_kF_k \\ 0 & A_k + B_kF_k \end{bmatrix},$$

is stable when A and $A_k + B_kF_k$ are stable, we know that $\left(\hat{A}, \begin{bmatrix} BD_k \\ B_k \end{bmatrix}\right)$ is stabilizable. Similarly, we can show that $\left(\hat{A}, \begin{bmatrix} D_kC & C_k \end{bmatrix}\right)$ is detectable. Therefore, if $\mathbf{Y} \in \mathcal{RH}_\infty$, we have A_{cl} is stable, meaning that $\mathbf{K} \in \mathcal{C}_{\text{stab}}$. This completes the proof. □

In the state-feedback case, we have the following result.

Corollary 2. *Consider the LTI system* (1)*. If* $C = I$*, then* $\mathbf{K} \in \mathcal{C}_{stab}$ *if and only if* $\left(\delta_x \to \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}\right) := \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \in \mathcal{RH}_\infty$.

The proof is similar by looking at the state-space representation. The result in Corollary 2 has been extensively used in the system-level synthesis [4]. The proof in [4] used a frequency-based method. Here, we provide an alternative proof from a state-space perspective.

4 Parameterization of stabilizing controllers

4.1 Two special cases

Corollary 1 leads to following parameterization of stabilizing controllers.

Corollary 3 ([7]). *Consider the LTI system* (1). *If the LTI system is open-loop stable, then we have*

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \middle| \begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{U} \end{bmatrix} = I,\ \mathbf{U} \in \mathcal{RH}_\infty \right\}.$$

Proof. $\Rightarrow$ Given any $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, we show there exist $\mathbf{Y}, \mathbf{U} \in \mathcal{RH}_\infty$ such that $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$ and the equality in the corollary is satisfied.

With $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, it is not difficult to derive

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{P}_{22}\mathbf{K})^{-1} \\ \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} \end{bmatrix} \delta_y.$$

Let us define $\mathbf{Y} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}$ and $\mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}$. Since $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, we know that $\mathbf{U} \in \mathcal{RH}_\infty$. Also, by definition, $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$. Finally, it is very easy to verify that

$$\mathbf{Y} - \mathbf{P}_{22}\mathbf{U} = (I - \mathbf{P}_{22}\mathbf{K})^{-1} - \mathbf{P}_{22}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} = I.$$

$\Leftarrow$ Given $\mathbf{Y}$ and $\mathbf{U}$ satisfying the condition, we show that $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{C}_{\text{stab}}$. By corollary 1, we only need to show the response from δ_y to $\mathbf{u}$ is Stable. In particular, we have

$$\begin{aligned} \mathbf{u} &= \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\delta_y \\ &= \mathbf{U}\mathbf{Y}^{-1}(I - \mathbf{P}_{22}\mathbf{U}\mathbf{Y}^{-1})^{-1}\delta_y \\ &= \mathbf{U}\delta_y, \end{aligned}$$

where the last equality used the affine relationship $\mathbf{Y} - \mathbf{P}_{22}\mathbf{U} = I$. □

This result is consistent with the classical one in [8, Theorem 12.7]. Open-loop stability of the plant does not provide a simplification for the SLP. Instead, if the state is directly measurable for control, *i.e.*, $C = I$, Corollary 2 leads to the following simplified SLP parameterization, which is denoted as the system-level parameterization in the state-feedback case [4, Theorem 1].

Corollary 4 ([4]). *Consider the LTI system* (1). *If* $C_2 = I$, *then we have*

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} = \mathbf{M}\mathbf{R}^{-1} \middle| \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I,\ \mathbf{M}, \mathbf{R} \in \mathcal{RH}_\infty \right\}.$$

The proof is very similar to that of Corollary 3.

4.2 General case: System-level parameterization and Input-output parameterization

It is not surprising that closed-loop responses are not independent to each other. In fact, they lie in a certain affine space. To be precise, given a stabilizing controller $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, the closed-loop responses $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ lie in the following affine subspace [3]

$$\begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{13a}$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{P}_{22} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{13b}$$

$$\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \in \mathcal{RH}_\infty. \tag{13c}$$

Further, we have the following result [3].

Theorem 1 (Input-output parameterization)**.** *The set of all internally stabilizing controllers can be represented as*

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \textit{ are in the affine subspace } (13a)\text{-}(13c)\}. \tag{14}$$

Proof. The proof is based on some straightforward algebra.

$\Rightarrow$: Given $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, we prove that there exist $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ in the affine space (13a)-(13c) such that $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$. In particular, we consider the closed-loop responses in (11), which are stable by definition. Then, it is easy to verify

$$\mathbf{Y} - \mathbf{P}_{22}\mathbf{U} = (I - \mathbf{P}_{22}\mathbf{K})^{-1} - \mathbf{P}_{22}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} = I, \tag{15}$$

and the rest of constraints in (13a) and (13b) are satisfied as well.

$\Leftarrow$: Given $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ in the affine space (13a)-(13c), we prove $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{C}_{\text{stab}}$. To do this, it is sufficient to check the closed-loop responses from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ are stable. For example, it is not difficult to show that

$$(I - \mathbf{P}_{22}\mathbf{K})^{-1} = (I - \mathbf{P}_{22}\mathbf{U}\mathbf{Y}^{-1})^{-1} = \mathbf{Y} \in \mathbf{RH}_\infty.$$

□

Similarly, given a stabilizing controller $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, the closed-loop responses $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ lie in the following affine subspace [4]

$$\begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{16a}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} sI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{16b}$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty. \tag{16c}$$

Theorem 2 (System-level parameterization)**.** *The set of all internally stabilizing controllers can be represented as*

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R},\ \mathbf{M},\ \mathbf{N},\ \mathbf{L} \textit{ are in the affine subspace } (16a)\text{-}(16c)\}. \tag{17}$$

The proof of Theorem 2 is very similar to that of Theorem 1. The interested reader is encouraged to verify the proof.

There are other equivalent parameterizations using different sets of closed-loop responses; see [6].

5 Convex reformulation of optimal controller synthesis

According to Theorem 1, the closed-loop response from w to z can be fully characterize by

$$\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21},$$

where $\mathbf{U}$ is in the affine space (13a)-(13c). Thus, the optimal controller synthesis (3) is equivalent to the following convex problem [3]

$$\begin{aligned} \min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (13\text{a}) - (13\text{c}). \end{aligned} \tag{18}$$

Similarly, we can derive that [4]

$$\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21} = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11},$$

and the optimal controller synthesis (3) is equivalent to the following convex problem

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & (16\text{a}) - (16\text{c}). \end{aligned} \tag{19}$$

6 Robust stability and its connection with learning applications

We provide some quick applications of learning-based control using Corollaries 1 and 2.

6.1 State feedback case

In the state feedback case, suppose we only have estimation $\hat{A}$ and $\hat{B}_2$, where $\|A - \hat{A}\| \leq \epsilon_A$ and $\|B - \hat{B}_2\| \leq \epsilon_B$. How can we design a stabilizing controller for the true system (A, B_2) based on the information $(\hat{A}, \hat{B}_2)$ and ϵ_A, ϵ_B?

Using Corollary 2, we find $\hat{\mathbf{M}}, \hat{\mathbf{R}} \in \mathcal{RH}_\infty$ that satisfies

$$\begin{bmatrix} sI - \hat{A} & -\hat{B}_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix} = I. \tag{20}$$

Then, the controller $\mathbf{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ stabilizes $(\hat{A}, \hat{B}_2)$. What happens if we apply $\mathbf{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ to the true system (A, B_2)?

From (21), we have

$$\begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix} = I + \mathbf{\Delta},$$

where $\mathbf{\Delta} = \Delta_A \hat{\mathbf{R}} + \Delta_B \hat{\mathbf{M}}$. Then it is not difficult to show that if $\|\Delta\|_\infty < 1$, the controller $\mathbf{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ stabilizes the true system (A, B_2) as well. This is one fundamental building block in the sample complexity and regret analysis of learning LQR controllers [1, 2].

6.2 Open-loop stable plants

In the open-loop stable case, we have similar results. First, suppose we have the transfer matrix estimation $\hat{\mathbf{P}}_{22}$, with $\|\mathbf{P}_{22} - \hat{\mathbf{P}}_{22}\|_\infty \leq \epsilon$.

Using corollary 1, we find $\hat{\mathbf{Y}}, \hat{\mathbf{U}} \in \mathcal{RH}_\infty$ that satisfies

$$\begin{bmatrix} I & -\hat{\mathbf{P}}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} \\ \hat{\mathbf{U}} \end{bmatrix} = I. \tag{21}$$

Then, the controller $\mathbf{K} = \hat{\mathbf{U}}\hat{\mathbf{Y}}^{-1}$ stabilizes the plant $\hat{\mathbf{P}}_{22}$. For the true plant $\mathbf{P}_{22}$, we have

$$\begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} \\ \hat{\mathbf{U}} \end{bmatrix} = I + \mathbf{\Delta},$$

where $\mathbf{\Delta} = \Delta \mathbf{P}_{22}$. Then it is not difficult to show that if $\|\Delta\|_\infty < 1$, the controller $\mathbf{K} = \hat{\mathbf{U}}\hat{\mathbf{Y}}^{-1}$ stabilizes the true system $\mathbf{P}_{22}$ as well.

7 Youla Parameterization

To add

An explicit equivalence among Theorem 1, Theorem 2, and the classical Youla parameterization [5] has been provided in [7].

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