Learning goals:

1. Youla parameterization for open-loop stable plants;
2. Disturbance feedback implementation and internal model principle;
3. Youla parameterization in finite-time horizon;
4. Doubly-coprime factorization and Youla
5. Equivalence with System-level synthesis, and input-output parameterization.

1 Recap

Consider a linear time-invariant system

\[ \dot{x} = Ax + B_2 u + \delta_x, \]
\[ y = C_2 x + \delta_y, \]  
(1)

and a dynamic output feedback controller \( u = Ky \), where \( K \) has a state-space realization

\[ \dot{\xi} = A_k \xi + B_k y, \]
\[ u = C_k \xi + D_k y, \]  
(2)

with \( \xi \in \mathbb{R}^{n_k} \) being the internal state of controller \( K \). We define the set of internally stabilizing controllers as

\[ C_{\text{stab}} := \{ K \mid K \text{ internally stabilizes } P \}, \]

and its state-space characterization is

\[ C_{\text{stab}} = \left\{ K \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable} \right\}, \]

where \( K = C_k (zI - A_k)^{-1} B_k + D_k \). We have introduced external transfer matrix characterizations of internal stability, and the corresponding system-level parameterization [9] and input-output parameterization [5] for \( C_{\text{stab}} \).

In this lecture, we present the classical Youla parameterization for \( C_{\text{stab}} \) [10], as well as a useful disturbance-based implementation. We also present an explicit equivalence among Youla, system-level, and input-output parameterizations [11].
2 Youla parameterization for open-loop stable plants

When the plant is open-loop stable, \( i.e., A \) is stable, then the Youla parameterization has a simple form.

**Theorem 1.** Suppose the plant is open-loop stable. Then, the set of all stabilizing controllers can be represented as

\[
C_{\text{stab}} = \{ K = Q(I + GQ)^{-1} | Q \in \mathcal{RH}_\infty \},
\]

where \( G = C_2(sI - A)^{-1}B_2 \).

**Proof.** \( \Rightarrow \): Suppose \( K_0 \in C_{\text{stab}} \). Then, we have \( Q_0 := K_0(I - GK_0)^{-1} \in \mathcal{RH}_\infty \) (which is the closed-loop response from \( \delta_y \) to \( u \)). It can be verified that \( K_0 \) can be expressed as follows

\[
Q_0(I + GQ_0)^{-1} = K_0(I - GK_0)^{-1}(I + GK_0(I - GK_0)^{-1})^{-1} = K_0.
\]

\( \Leftarrow \): Suppose \( Q \in \mathcal{RH}_\infty \), and define \( K = Q(I + GQ)^{-1} \). We verify this controller internally stabilizes the plant. Since the plant is open-loop stable, we only need to check the closed-loop response from \( \delta_y \) to \( u \) is stable.

\[
u = K(I - GK)^{-1}\delta_y
= Q(I + GQ)^{-1}(I - GQ(I + GQ)^{-1})^{-1}\delta_y
= Q\delta_y.
\]

This completes the proof. \( \Box \)

From the proof above, it is easy to see that the Youla parameter \( Q \) is exactly the same as the closed-loop response from \( \delta_y \) to \( u \). This is identical to the input-output parameterization [5].

2.1 Disturbance feedback implementation

The controller \( K = Q(I + GQ)^{-1} \) can be implemented in a disturbance-based form (see Figure 1 for illustration):

\[
\beta = y - Gu,
\]

\[
u = Q\beta.
\]  \hspace{1cm} (4)

Recall that there is measurement noise in the plant dynamics, \( i.e., y = Gu + \delta_y \). Thus, if there is no noise in the control input, then in (4), we have \( \beta = \delta_y \), and

\[
u = Q\delta_y,
\]

which is a disturbance feedback implementation. Note that \( \delta_y \) is referred to as “nature’s \( y \)” in [8].

 Especially, in the discrete time, when the plant \( G \) is strictly proper and approximated by a finite impulse response with length \( p \) and the Youla parameter is approximated by a finite impulse response with length \( q \), \( i.e., \)

\[
G = \sum_{k=1}^{p} G_k \frac{1}{z^k}, \quad Q = \sum_{k=0}^{q} Q_k \frac{1}{z^k},
\]

then (4) can be implemented as

\[
\beta_t = y_t - \sum_{k=1}^{p} G_k u_{t-k},
\]

\[
u_t = \sum_{k=0}^{q} Q_k \beta_{t-k}.
\]
Figure 1: Internal model principle, where $P_{22} := G$.

This disturbance-based implementation is explicitly used in [8] for regret analysis.

**Internal model principle:** In Fig. 1, we note that the controller $K$ explicitly incorporates the plant dynamics $G$, which is known as the internal model principle [4] applied in Youla parameterization. The following paragraph is quoted from [2]: “The concept of internal models plays a crucial role in regulator problems. The internal model principle can intuitively be expressed as: ‘Any good regulator must create a model of the dynamic structure of the environment in the closed loop system’.”

### 3 Youla parameterization in finite-time horizon

In this section, we discuss the Youla parameterization in the finite-time horizon. The disturbance-based parameterization allows us to get a convex characterization of time-varying feedback policies with constraints on state and inputs. For simplicity, we consider state feedback policies in this section. The presentation of this section is based on [6].

Consider the following discrete-time LTI system:

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^n$ is the disturbance at the current time instant. The system is subject to mixed constraints on the state and input:

$$Z := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\},$$

where the matrices $C \in \mathbb{R}^{s \times n}, D \in \mathbb{R}^{s \times m}$ and the vector $b \in \mathbb{R}^s$. It is assumed that $Z$ is bounded and contains the origin in its interior. A primary design goal is to guarantee that the state and input of the closed-loop system remain in $Z$ for all time and for all allowable disturbance sequences. Finally, a target/terminal constraint set $X_f$ is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\},$$

where the matrix $Y \in \mathbb{R}^{r \times n}$ and the vector $z \in \mathbb{R}^r$. It is assumed that $X_f$ is bounded and contains the origin in its interior.

In the sequel, predictions of the system’s evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length $N$ of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $u \in \mathbb{R}^{mN}, x \in \mathbb{R}^{n(N+1)}$ and $w \in \mathbb{R}^{nN}$, respectively, as

$$x := \begin{bmatrix} x_0^T, \ldots, x_N^T \end{bmatrix}^T,$$

$$u := \begin{bmatrix} u_0^T, \ldots, u_{N-1}^T \end{bmatrix}^T,$$

$$w := \begin{bmatrix} w_0^T, \ldots, w_{N-1}^T \end{bmatrix}^T,$$
where $x_0 = x$ denotes the current measured value of the state. Finally, let the set $W := W \times \ldots \times W$, so that $w \in W$.

Then, the system can be compactly written as

$$ x = Ax + Bu + Ew, $$

where

$$ A = \begin{bmatrix} I & 0 & 0 & \ldots & 0 \\ A & 0 & 0 & \ldots & 0 \\ 0 & A & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ B & 0 & \ldots & 0 \\ 0 & B & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \ldots & 0 & B \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ I & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & \ldots & 0 & I \end{bmatrix}. $$

**State feedback parameterization:** One natural approach to controlling the system in (5), while ensuring the satisfaction of the constraints, is to search over the set of time-varying affine state feedback control policies with knowledge of prior states:

$$ u_t = \sum_{i=0}^{t} L_{t,i} x_i + g_t, \quad t = 0, \ldots, N - 1, \quad (8) $$

where each $L_{t,i} \in \mathbb{R}^{m \times n}$ and $g_t \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $L \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $g \in \mathbb{R}^{mN}$ as

$$ L = \begin{bmatrix} L_{0,0} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N,0} & \ldots & L_{N,N-1} & 0 \end{bmatrix}, \quad g = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}. \quad (9) $$

Then, the input sequence can be written as

$$ u = Lx + g. $$

For a given initial state $x$, we say that the pair $(L, g)$ is admissible if the control policy (8) guarantees that for all allowable disturbance sequences of length $N$, the constraints (6) are satisfied over the horizon $t = 0, \ldots, N - 1$ and that the state is in the target set (7) at the end of the horizon. Precisely, the set of admissible $(L, g)$ is defined as

$$ \Pi_{sf}^N(x) := \left\{ (L, g) \mid (L, g) \text{ satisfy (9)}, x_0 = x, \\
\text{s.t. } x_{t+1} = Ax_t + Bu_t + w_t \right\}. \quad (10) $$

**Proposition 1** ([6]). The set of admissible affine state feedback parameters $\Pi_{sf}^N(x)$ is non-convex.

**Disturbance feedback parameterization:** An alternative to (8) is to parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$ u_t = \sum_{i=0}^{t-1} M_{t,i} w_t + v_t, \quad t = 0, \ldots, N - 1 \quad (11) $$
where each $M_{t,i} \in \mathbb{R}^{m \times n}$ and $v_t \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance sequence is easily calculated as the difference between the predicted and actual states at each step, i.e.

$$w_{t-1} = x_t - Ax_{t-1} - Bu_{t-1}.$$  

For notational convenience, we define the vector $v_t \in \mathbb{R}^m$ and the strictly block lower triangular matrix $M \in \mathbb{R}^{mN \times nN}$ such that

$$M = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \quad v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}. \tag{12}$$

Then, the input sequence can be written as

$$u = Mw + v.$$  

In a manner similar to (10), we define the set of admissible $(M, v)$ as

$$\Pi^{df}_N(x) := \left\{ (M, v) \mid (M, v) \text{ satisfy (12)}, x_0 = x, \right. \right. \left. \left. x_{t+1} = Ax_t + Bu_t + w_t \right. \right. \left. \right. \right.$$  

$$u_t = \sum_{i=0}^{t-1} M_{t,i} w_i + v_t, \quad (x_t, u_t) \in \mathcal{Z}, \quad x_N \in \mathcal{X},$$  

$$t = 0, \ldots, N-1, \forall w \in \mathcal{W}.$$

It can be checked that one can find matrices $F \in \mathbb{R}^{l \times mN}, G \in \mathbb{R}^{l \times nN}, H \in \mathbb{R}^{l \times n}$ and a vector $c \in \mathbb{R}^s$, where $l := sN + r$ (see the Appendix of [6]), such that the expression for $\Pi^{df}_N(x)$ can be rewritten more compactly as

$$\Pi^{df}_N(x) := \left\{ (M, v) \mid (M, v) \text{ satisfy (12)} \right. \right. \left. \right. \right.$$  

$$Fv + (FM + G)w \leq c + Hx, \quad \forall w \in \mathcal{W}.$$  

We note that this is easily seen from the fact that under the policy (11), the state and input sequences can be written as

$$x = (I - A)^{-1}(BM + E)w + (I - A)^{-1}Bv,$$

$$u = Mw + v. \tag{14}$$

Then it is easy to see the following result.

**Proposition 2.** The set of admissible affine disturbance feedback parameters $\Pi^{df}_N(x)$ is convex.

Also, we have the following equivalence.

**Theorem 2 ([6]).** For any admissible $(L, g)$, an admissible $(M, v)$ can be found that yields the same state and input sequence for all allowable disturbance sequences, and vice-versa.

**Proof.** $\Rightarrow$ Given $(L, g)$, we find $(M, v)$ that yields the same state and input sequence. First, we have

$$x = Ax + B(Lx + g) + Ew$$

$$\Rightarrow x = (I - A - BL)^{-1}Ew + (I - A - BL)^{-1}Bg$$

$$\Rightarrow u = L(I - A - BL)^{-1}Ew + L(I - A - BL)^{-1}Bg + g$$
Let us define
\[ M = L(I - A - BL)^{-1}E, \quad v = L(I - A - BL)^{-1}Bg + g, \]
then, the closed-loop system with \((M, v)\) yields the same state and input sequence. It is routinely to show that \((M, v)\) has the same structure in (12).

\[ \Leftarrow \text{Almost similar; see [6] for details.} \]

We note that the disturbance feedback implementation has recently been employed in online learning with adversarial disturbances in [1].

4 Doubly co-prime factorization and Youla parameterization

Here, we introduce the Youla parameterization for general plants, which is based on a doubly coprime factorization.

**Definition 1.** A collection of stable transfer matrices, \(U_l, V_l, N_l, M_l, U_r, V_r, N_r, M_r \in \mathcal{RH}_\infty\) is called a doubly-coprime factorization of \(G\) if
\[ G = N_r M_r^{-1} = M_l^{-1} N_l \quad \text{and} \quad \begin{bmatrix} U_l & -V_l \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & V_r \\ N_r & U_r \end{bmatrix} = I. \]

Such doubly-coprime factorization can always be computed if the state-space realization of \(G\) is stabilizable and detectable [7]. We have the following equivalence [10]
\[ C_{\text{stab}} = \{ K = (V_r - M_r Q)(U_r - N_r Q)^{-1} \mid Q \in \mathcal{RH}_\infty \}, \quad (15) \]
where \(Q\) is denoted as the Youla parameter. Note that the Youla parameter \(Q\) can be freely chosen in \(\mathcal{RH}_\infty\). We refer the interested reader to [3,10,12] for more details on the Youla parameterization. Note that it is not difficult to derive a convex reformulation of the original optimal control problem in terms of the Youla parameter: Using the change of variables \(K = (V_r - M_r Q)(U_r - N_r Q)^{-1}\), one can derive
\[ f(P, K) = T_{11} + T_{12} Q T_{21}, \]
where \(T_{11} = P_{11} + P_{12} V_r M_l P_{21}, T_{12} = -P_{12} M_r,\) and \(T_{21} = M_l P_{21}\). Consequently, the optimal control problem can be equivalently reformulated in terms of the Youla parameter as
\[ \min_Q \| T_{11} + T_{12} Q T_{21} \| \]
subject to \( Q \in \mathcal{RH}_\infty \).

**Computation of doubly-coprime factorization:** It is numerically easy to find a doubly coprime factorization if the plant is stabilizable and detectable [12, Theorem 5.9].

**Theorem 3.** Suppose \(G(s)\) is a proper real-rational matrix and
\[ G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]
is a stabilizable and detectable realization. Let $F$ and $L$ be such that $A + BF$ and $A + LC$ are both stable, and a doubly co-prime factorization of $G$ is as follows.

$$
\begin{bmatrix}
M_r & V_r \\
N_r & U_r
\end{bmatrix} = \begin{bmatrix}
\begin{array}{cc}
A + BF & B \\
F & I \\
C + DF & D \\
\end{array}
\end{bmatrix},
$$

(17)

$$
\begin{bmatrix}
U_l & -V_l \\
-N_l & M_l
\end{bmatrix} = \begin{bmatrix}
\begin{array}{cc}
A + LC & -(B + LD) \\
F & I \\
C & -D \\
\end{array}
\end{bmatrix},
$$

Proof. It is based on directly verification. See [7] for details.

Feedback control interpretation: The coprime factorization of a transfer matrix can be given a feedback control interpretation. For example, right coprime factorization comes out naturally from changing the control variable by a state feedback. Consider the state-space model

$$
\dot{x} = Ax + Bu,
$$

$$
y = Cx + Du.
$$

Next, introduce a state feedback and change the variable

$$
v := u - Fx
$$

where $F$ is such that $A + BF$ is stable. Then, we get

$$
\dot{x} = (A + BF)x + Bv,
$$

$$
u = Fx + v
$$

$$
y = (C + DF)x + Dv.
$$

From these equations, the transfer matrix from $v$ to $u$ is

$$
M_r(s) = \begin{bmatrix}
A + BF & B \\
F & I
\end{bmatrix},
$$

and that from $v$ to $y$ is

$$
N_r(s) = \begin{bmatrix}
A + BF & B \\
C + DF & D
\end{bmatrix}.
$$

Therefore, we have

$$
u = M_r v, \quad y = N_r v,
$$

so that $y = N_r M_r^{-1} u$, i.e., $G = N_r M_r^{-1}$.

5 Equivalence with SLP and IOP

An explicit equivalence among Youla, the SLP, and the IOP has been recently revealed in [11].
References


