Optimal control and Convex optimization **Spring 2020**

Lecture 3: Youla Parameterization

Author: Yang Zheng

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They were developed when the author was a postdoc in Prof. Na Li's group at Harvard. Any typos should be sent to* `zhengy@eng.ucsd.edu`.

Learning goals:

1. Youla parameterization for open-loop stable plants;
2. Disturbance feedback implementation and internal model principle;
3. Youla parameterization in finite-time horizon;
4. Doubly-coprime factorization and Youla
5. Equivalence with System-level synthesis, and input-output parameterization.

1 Recap

Consider a linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + B_2 u + \delta_x, \\ y &= C_2 x + \delta_y, \end{aligned} \tag{1}$$

and a dynamic output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ has a state-space realization

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y, \\ u &= C_k \xi + D_k y, \end{aligned} \tag{2}$$

with $\xi \in \mathbb{R}^{n_k}$ being the internal state of controller $\mathbf{K}$. We define the set of internally stabilizing controllers as

$$\mathcal{C}_{\text{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{P}\},$$

and its state-space characterization is

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{is stable} \right\},$$

where $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. We have introduced external transfer matrix characterizations of internal stability, and the corresponding system-level parameterization [9] and input-output parameterization [5] for $\mathcal{C}_{\text{stab}}$.

In this lecture, we present the classical Youla parameterization for $\mathcal{C}_{\text{stab}}$ [10], as well as a useful disturbance-based implementation. We also present an explicit equivalence among Youla, system-level, and input-output parameterizations [11].

2 Youla parameterization for open-loop stable plants

When the plant is open-loop stable, *i.e.*, A is stable, then the Youla parameterization has a simple form.

Theorem 1. *Suppose the plant is open-loop stable. Then, the set of all stabilizing controllers can be represented as*

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\}, \tag{3}$$

where $\mathbf{G} = C_2(sI - A)^{-1}B_2$*.*

Proof. $\Rightarrow$: Suppose $\mathbf{K}_0 \in \mathcal{C}_{\text{stab}}$. Then, we have $\mathbf{Q}_0 := \mathbf{K}_0(I - \mathbf{G}\mathbf{K}_0)^{-1} \in \mathcal{RH}_\infty$ (which is the closed-loop response from δ_y to $\mathbf{u}$). It can be verified that $\mathbf{K}_0$ can be expressed as follows

$$\mathbf{Q}_0(I + \mathbf{G}\mathbf{Q}_0)^{-1} = \mathbf{K}_0(I - \mathbf{G}\mathbf{K}_0)^{-1}(I + \mathbf{G}\mathbf{K}_0(I - \mathbf{G}\mathbf{K}_0)^{-1})^{-1} = \mathbf{K}_0.$$

$\Leftarrow$: Suppose $\mathbf{Q} \in \mathcal{RH}_\infty$, and define $\mathbf{K} = \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1}$. We verify this controller internally stabilizes the plant. Since the plant is open-loop stable, we only need to check the closed-loop response from δ_y to $\mathbf{u}$ is stable.

$$\begin{aligned}\mathbf{u} &= \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\delta_y \\ &= \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1}(I - \mathbf{G}\mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1})^{-1}\delta_y \\ &= \mathbf{Q}\delta_y.\end{aligned}$$

This completes the proof. □

From the proof above, it is easy to see that the Youla parameter $\mathbf{Q}$ is exactly the same as the closed-loop response from δ_y to $\mathbf{u}$. This is identical to the input-output parameterization [5].

2.1 Disturbance feedback implementation

The controller $\mathbf{K} = \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1}$ can be implemented in a disturbance-based form (see Figure 1 for illustration):

$$\begin{aligned}\beta &= \mathbf{y} - \mathbf{G}\mathbf{u}, \\ \mathbf{u} &= \mathbf{Q}\beta.\end{aligned} \tag{4}$$

Recall that there is measurement noise in the plant dynamics, *i.e.*, $\mathbf{y} = \mathbf{G}\mathbf{u} + \delta_y$. Thus, if there is no noise in the control input, then in (4), we have $\beta = \delta_y$, and

$$\mathbf{u} = \mathbf{Q}\delta_y,$$

which is a disturbance feedback implementation. Note that δ_y is referred to as "nature's y" in [8]. Especially, in the discrete time, when the plant $\mathbf{G}$ is strictly proper and approximated by a finite impulse response with length p and the Youla parameter is approximated by a finite impulse response with length q, *i.e.*,

$$\mathbf{G} = \sum_{k=1}^{p} G_k \frac{1}{z^k}, \qquad \mathbf{Q} = \sum_{k=0}^{q} Q_k \frac{1}{z^k},$$

then (4) can be implemented as

$$\begin{aligned}\beta_t &= y_t - \sum_{k=1}^{p} G_k u_{t-k}, \\ u_t &= \sum_{k=0}^{q} Q_k \beta_{t-k}.\end{aligned}$$

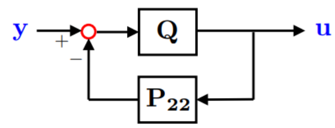

Figure 1: Internal model principle, where $\mathbf{P}_{22} := \mathbf{G}$.

This disturbance-based implementation is explicitly used in [8] for regret analysis.

Internal model principle: In Fig. 1, we note that the controller $\mathbf{K}$ explicitly incorporates the plant dynamics $\mathbf{G}$, which is known as the internal model principle [4] applied in Youla parameterization. The following paragraph is quoted from [2]: "*The concept of internal models plays a crucial role in regulator problems. The internal model principle can intuitively be expressed as: 'Any good regulator must create a model of the dynamic structure of the environment in the closed loop system'* ".

3 Youla parameterization in finite-time horizon

In this section, we discuss the Youla parameterization in the finite-time horizon. The disturbance-based parameterization allows us to get a convex characterization of time-varying feedback policies with constraints on state and inputs. For simplicity, we consider state feedback policies in this section. The presentation of this section is based on [6].

Consider the following discrete-time LTI system:

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{5}$$

where $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^n$ is the disturbance at the current time instant. The system is subject to mixed constraints on the state and input:

$$\mathcal{Z} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\}, \tag{6}$$

where the matrices $C \in \mathbb{R}^{s\times n}, D \in \mathbb{R}^{s\times m}$ and the vector $b \in \mathbb{R}^s$. It is assumed that Z is bounded and contains the origin in its interior. A primary design goal is to guarantee that the state and input of the closed-loop system remain in $\mathcal{Z}$ for all time and for all allowable disturbance sequences. Finally, a target/terminal constraint set X_f is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{7}$$

where the matrix $Y \in \mathbb{R}^{r\times n}$ and the vector $z \in \mathbb{R}^r$. It is assumed that X_f is bounded and contains the origin in its interior.

In the sequel, predictions of the system's evolution over a finite control/planning horizon will be used to define a number of suitable control policies. Let the length N of this planning horizon be a positive integer and define stacked versions of the predicted input, state and disturbance vectors $u \in \mathbb{R}^{mN}, x \in \mathbb{R}^{n(N+1)}$ and $w \in \mathbb{R}^{nN}$, respectively, as

$$\begin{aligned}
\mathbf{x} &:= \left[x_0^\top, \dots, x_N^\top\right]^\top, \\
\mathbf{u} &:= \left[u_0^\top, \dots, u_{N-1}^\top\right]^\top, \\
\mathbf{w} &:= \left[w_0^\top, \dots, w_{N-1}^\top\right]^\top,
\end{aligned}$$

where $x_0 = x$ denotes the current measured value of the state. Finally, let the set $\mathcal{W} := W \times \ldots \times W$, so that $\mathbf{w} \in \mathcal{W}$.

Then, the system can be compactly written as

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w},$$

where

$$\mathbf{A} = \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ A & 0 & 0 & \dots & 0 \\ 0 & A & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & B \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I \end{bmatrix}.$$

State feedback parameterization: One natural approach to controlling the system in (5), while ensuring the satisfaction of the constraints, is to search over the set of time-varying affine state feedback control policies with knowledge of prior states:

$$u_t = \sum_{i=0}^{t} L_{t,i} x_i + g_t, \qquad t = 0, \dots, N-1, \tag{8}$$

where each $L_{t,i} \in \mathbb{R}^{m \times n}$ and $g_t \in \mathbb{R}^m$. For notational convenience, we also define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} = \begin{bmatrix} L_{0,0} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N_1,0} & \dots & L_{N_1,N-1} & 0 \end{bmatrix}, \mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}. \tag{9}$$

Then, the input sequence can be written as

$$\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}.$$

For a given initial state x, we say that the pair $(\mathbf{L}, \mathbf{g})$ is admissible if the control policy (8) guarantees that for all allowable disturbance sequences of length N, the constraints (6) are satisfied over the horizon $t = 0, \dots, N-1$ and that the state is in the target set (7) at the end of the horizon. Precisely, the set of admissible $(\mathbf{L}, \mathbf{g})$ is defined as

$$\Pi_N^{\text{sf}}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \;\middle|\; \begin{array}{l} (\mathbf{L}, \mathbf{g}) \text{ satisfy (9)}, x_0 = x, \\ x_{t+1} = A x_t + B u_t + w_t \\ u_t = \sum_{i=0}^{t} L_{t,i} x_i + g_t \\ (x_t, u_t) \in Z, x_N \in X_f \\ t = 0, \dots, N-1, \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{10}$$

Proposition 1 ([6])**.** *The set of admissible affine state feedback parameters $\Pi_N^{sf}(x)$ is non-convex.*

Disturbance feedback parameterization: An alternative to (8) is to parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$u_t = \sum_{i=0}^{t-1} M_{t,i} w_t + v_t, \qquad t = 0, \dots, N-1 \tag{11}$$

where each $M_{t,i} \in \mathbb{R}^{m\times n}$ and $v_t \in \mathbb{R}^m$. It should be noted that, since full state feedback is assumed, the past disturbance sequence is easily calculated as the difference between the predicted and actual states at each step, i.e.

$$w_{t-1} = x_t - Ax_{t-1} - Bu_{t-1}.$$

For notational convenience, we define the vector $\mathbf{v} \in \mathbf{R}^{mN}$ and the strictly block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN\times nN}$ such that

$$\mathbf{M} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}. \tag{12}$$

Then, the input sequence can be written as

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}.$$

In a manner similar to (10), we define the set of admissible $(\mathbf{M}, \mathbf{v})$ as

$$\Pi_N^{\mathrm{df}}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \,\middle|\, \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfy } (12), x_0 = x, \\ x_{t+1} = Ax_t + Bu_t + w_t \\ u_t = \sum_{i=0}^{t-1} M_{t,i} w_i + v_t \\ (x_t, u_t) \in Z, x_N \in X_f \\ t = 0, \ldots, N-1, \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}. \tag{13}$$

It can be checked that one can find matrices $F \in \mathbb{R}^{l\times mN}, G \in \mathbb{R}^{l\times nN}, H \in \mathbb{R}^{l\times n}$ and a vector $c \in \mathbb{R}^s$, where $l := sN + r$ (see the Appendix of [6]), such that the expression for $\Pi_N^{\mathrm{df}}(x)$ can be rewritten more compactly as

$$\Pi_N^{\mathrm{df}}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \,\middle|\, \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfy } (12) \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \le c + Hx, \\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}.$$

We note that this is easily seen from the fact that under the policy (11), the state and input sequences can be written as

$$\begin{aligned} \mathbf{x} &= (I - \mathbf{A})^{-1}(\mathbf{B}\mathbf{M} + \mathbf{E})\mathbf{w} + (I - \mathbf{A})^{-1}\mathbf{B}\mathbf{v}, \\ \mathbf{u} &= \mathbf{M}\mathbf{w} + \mathbf{v}. \end{aligned} \tag{14}$$

Then it is easy to see the following result.

Proposition 2. *The set of admissible affine disturbance feedback parameters $\Pi_N^{df}(x)$ is convex.*

Also, we have the following equivalence.

Theorem 2 ([6]). *For any admissible $(\mathbf{L}, \mathbf{g})$, an admissible $(\mathbf{M}, \mathbf{v})$ can be found that yields the same state and input sequence for all allowable disturbance sequences, and vice-versa.*

Proof. $\Rightarrow$ Given $(\mathbf{L}, \mathbf{g})$, we find $(\mathbf{M}, \mathbf{v})$ that yields the same state and input sequence. First, we have

$$\begin{aligned} &\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{L}\mathbf{x} + \mathbf{g}) + \mathbf{E}\mathbf{w} \\ \Rightarrow &\mathbf{x} = (I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w} + (I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{B}\mathbf{g} \\ \Rightarrow &\mathbf{u} = \mathbf{L}(I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w} + \mathbf{L}(I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{B}\mathbf{g} + \mathbf{g} \end{aligned}$$

Let us define

$$\mathbf{M} = \mathbf{L}(I - \mathbf{A} - \mathbf{BL})^{-1}\mathbf{E}, \qquad \mathbf{v} = \mathbf{L}(I - \mathbf{A} - \mathbf{BL})^{-1}\mathbf{Bg} + \mathbf{g},$$

then, the closed-loop system with $(\mathbf{M}, \mathbf{v})$ yields the same state and input sequence. It is routinely to show that $(\mathbf{M}, \mathbf{v})$ has the same structure in (12).

$\Leftarrow$: Almost similar; see [6] for details. □

We note that the disturbance feedback implementation has recently been employed in online learning with adversarial disturbances in [1].

4 Doubly co-prime factorization and Youla parameterization

Here, we introduce the Youla parameterization for general plants, which is based on a doubly coprime factorization.

Definition 1. *A collection of stable transfer matrices,* $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r \in \mathcal{RH}_\infty$ *is called a doubly-coprime factorization of* $\mathbf{G}$ *if* $\mathbf{G} = \mathbf{N}_r\mathbf{M}_r^{-1} = \mathbf{M}_l^{-1}\mathbf{N}_l$ *and*

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = I.$$

Such doubly-coprime factorization can always be computed if the state-space realization of $\mathbf{G}$ is stabilizable and detectable [7]. We have the following equivalence [10]

$$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\}, \tag{15}$$

where $\mathbf{Q}$ is denoted as the Youla parameter. Note that the Youla parameter $\mathbf{Q}$ can be freely chosen in $\mathcal{RH}_\infty$. We refer the interested reader to [3,10,12] for more details on the Youla parameterization. Note that it is not difficult to derive a convex reformulation of the original optimal control problem in terms of the Youla parameter: Using the change of variables $\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}$, one can derive

$$f(\mathbf{P}, \mathbf{K}) = \mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21},$$

where $\mathbf{T}_{11} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{V}_r\mathbf{M}_l\mathbf{P}_{21}, \mathbf{T}_{12} = -\mathbf{P}_{12}\mathbf{M}_r$, and $\mathbf{T}_{21} = \mathbf{M}_l\mathbf{P}_{21}$. Consequently, the optimal control problem can be equivalently reformulated in terms of the Youla parameter as

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty. \end{aligned} \tag{16}$$

Computation of doubly-coprime factorization: It is numerically easy to find a doubly coprime factorization if the plant is stabilizable and detectable [12, Theorem 5.9].

Theorem 3. *Suppose* $\mathbf{G}(s)$ *is a proper real-rational matrix and*

$$\mathbf{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

is a stabilizable and detectable realization. Let F and L be such that $A+BF$ and $A+LC$ are both stable, and a doubly co-prime factorization of $\mathbf{G}$ *is as follows.*

$$\begin{aligned}\begin{bmatrix}\mathbf{M}_r & \mathbf{V}_r\\ \mathbf{N}_r & \mathbf{U}_r\end{bmatrix} &= \left[\begin{array}{c|cc} A+BF & B & L\\ \hline F & I & 0\\ C+DF & D & I\end{array}\right],\\ \begin{bmatrix}\mathbf{U}_l & -\mathbf{V}_l\\ -\mathbf{N}_l & \mathbf{M}_l\end{bmatrix} &= \left[\begin{array}{c|cc} A+LC & -(B+LD) & L\\ \hline F & I & 0\\ C & -D & I\end{array}\right],\end{aligned} \tag{17}$$

Proof. It is based on directly verification. See [7] for details. □

Feedback control interpretation: The coprime factorization of a transfer matrix can be given a feedback control interpretation. For example, right coprime factorization comes out naturally from changing the control variable by a state feedback. Consider the state-space model

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx + Du.\end{aligned}$$

Next, introduce a state feedback and change the variable

$$v := u - Fx$$

where F is such that $A+BF$ is stable. Then, we get

$$\begin{aligned}\dot{x} &= (A+BF)x + Bv,\\ u &= Fx + v\\ y &= (C+DF)x + Dv.\end{aligned}$$

From these equations, the transfer matrix from v to u is

$$\mathbf{M}_r(s) = \left[\begin{array}{c|c} A+BF & B\\ \hline F & I\end{array}\right],$$

and that from v to y is

$$\mathbf{N}_r(s) = \left[\begin{array}{c|c} A+BF & B\\ \hline C+DF & d\end{array}\right].$$

Therefore, we have

$$\mathbf{u} = \mathbf{M}_r\mathbf{v}, \qquad \mathbf{y} = \mathbf{N}_r\mathbf{v},$$

so that $\mathbf{y} = \mathbf{N}_r\mathbf{M}_r^{-1}\mathbf{u}$, *i.e.*, $\mathbf{G} = \mathbf{N}_r\mathbf{M}_r^{-1}$.

5 Equivalence with SLP and IOP

An explicit equivalence among Youla, the SLP, and the IOP has been recently revealed in [11].

References

[1] Naman Agarwal, Brian Bullins, Elad Hazan, Sham M Kakade, and Karan Singh. Online control with adversarial disturbances. *arXiv preprint arXiv:1902.08721*, 2019.

[2] Gunnar Bengtsson. Output regulation and internal models—a frequency domain approach. *Automatica*, 13(4):333–345, 1977.

[3] A. Francis. *A course in $\mathcal{H}_\infty$ control theory*. Springer-Verlag, 1987.

[4] Bruce A Francis and Walter Murray Wonham. The internal model principle of control theory. *Automatica*, 12(5):457–465, 1976.

[5] Luca Furieri, Yang Zheng, Antonis Papachristodoulou, and Maryam Kamgarpour. An input-output parametrization of stabilizing controllers: amidst youla and system level synthesis. *IEEE Control Systems Letters*, 3(4):1014–1019, Oct 2019.

[6] Paul J Goulart, Eric C Kerrigan, and Jan M Maciejowski. Optimization over state feedback policies for robust control with constraints. *Automatica*, 42(4):523–533, 2006.

[7] C Nett, C Jacobson, and M Balas. A connection between state-space and doubly coprime fractional representations. *IEEE Transactions on Automatic Control*, 29(9):831–832, 1984.

[8] Max Simchowitz, Karan Singh, and Elad Hazan. Improper learning for non-stochastic control. *arXiv preprint arXiv:2001.09254*, 2020.

[9] Yuh-Shyang Wang, Nikolai Matni, and John C Doyle. A system level approach to controller synthesis. *IEEE Transactions on Automatic Control*, 2019.

[10] Dante Youla, Hamid Jabr, and Jr Bongiorno. Modern wiener-hopf design of optimal controllers–part ii: The multivariable case. *IEEE Transactions on Automatic Control*, 21(3):319–338, 1976.

[11] Yang Zheng, Luca Furieri, Antonis Papachristodoulou, Na Li, and Maryam Kamgarpour. On the equivalence of youla, system-level and input-output parameterizations. *arXiv preprint arXiv:1907.06256*, 2019.

[12] Kemin Zhou, John Comstock Doyle, Keith Glover, et al. *Robust and optimal control*, volume 40. Prentice hall New Jersey, 1996.