Optimal control and Convex optimization **Spring 2020**

Lecture 4: LMI formulation for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They were developed when the author was a postdoc in Prof. Na Li's group at Harvard. Any typos should be sent to zhengy@eng.ucsd.edu.*

Learning goals:

1. Hardy Spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$;
2. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms and their state-space computations;
3. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control: state feedback case
4. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control: output feedback case

1 Recap

The problem setup is as follows: we consider continuous-time linear time-invariant (LTI) systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^d, y \in \mathbb{R}^p, z \in \mathbb{R}^q$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively. Consider a dynamic output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ has a state-space realization

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y, \\ u &= C_k \xi + D_k y, \end{aligned} \tag{2}$$

where $\xi \in \mathbb{R}^{n_k}$ is the internal state of controller $\mathbf{K}$.

We have introduced the following optimal control problem

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}, \end{aligned} \tag{3}$$

and its corresponding state-space version (where we have assumed that $D_{22} = 0$) is

$$\begin{aligned} \min_{A_k, B_k, C_k, D_k} \quad & \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\| \\ \text{subject to} \quad & \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.} \end{aligned} \tag{4}$$

Both (3) and (4) are non-convex in its present form. Note that the formulation (3) or (4) is very general, including LQR/LQG/$\mathcal{H}_2$/$\mathcal{H}_\infty$ optimal control [5].

In Lectures 2 and 3, we have presented convex reformulation of (3) by introducing a suitable change of variables (*i.e.*, Youla, IOP and SLP) in the frequency domain. In this lecture, we will take a closer look at the cost function and its state-space solution via solving linear matrix inequalities in both state feedback and output feedback cases.

2 Hardy Spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$

The presentation of this section largely follows [5, Section 4.3]. Let $S \subset \mathbb{C}$ be an open set, and let $f(s)$ be a complex valued function defined on S:

$$f(s) : S \to \mathbb{C}.$$

Then, $f(s)$ is said to be analytic at a point z_0 in S if it is differentiable at z_0 and also at each point in some neighborhood of z_0. The following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

In fact, if $f(s)$ is analytic at z_0 then f has continuous derivatives of all orders at z_0. Hence, a function $f(s)$ analytic at z_0 has a power serices representation at z_0, *i.e.*,

$$f(s) = c_0 + \sum_{n=1}^{\infty} c_n (s - z_0)^n,$$

converges for some neighborhood of z_0. The converse is also true, *i.e.*, if a function has a power series representation at z_0, then it is analytic at z_0.

A function $f(s)$ is said to be analytic in S if it has a derivative or is analytic at each point of S. A matrix valued function is analytic in S if every element of the matrix is analytic in S. For example, all real rational stable transfer matrices are analytic in the right-half plane and e^{-s} is analytic everywhere. A well-known property of the analytic functions is the so-called *Maximum Modulus Theorem*.

Theorem 1. *If $f(s)$ is defined and continuous on a closed-bounded set S and analytic on the interior of S, then the maximum of $|f(s)|$ on S is attained on the boundary of S,* i.e.*,*

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|,$$

where ∂S denotes the boundary of S.

Next we consider some frequently used complex (matrix) function spaces.

1. $\mathcal{L}_2(j\mathbb{R})$ **Space:** $\mathcal{L}_2(j\mathbb{R})$ is a Hilbert space of matrix-valued function on $j\mathbb{R}$ and consists of all complex matrix functions F such that the integral below is bounded, *i.e.*,

$$\int_{-\infty}^{\infty} \operatorname{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega < \infty.$$

The inner product for this Hilbert space is defined as

$$\langle F, G\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)G(j\omega)\right] d\omega < \infty,$$

for $F, G \in \mathcal{L}_2$, and the inner product induced norm is given by

$$\|F\|_2 := \sqrt{\langle F, F\rangle}.$$

All real rational strictly proper transfer functions with no poles on the imaginary axis form a subspace of $\mathcal{L}_2$. which is denoted by $\mathcal{RL}_2$.

2. $\mathcal{H}_2$ **Space:** $\mathcal{H}_2$ is subspace of $\mathcal{L}_2$ with matrix functions $F(s)$ analytic in $\text{Re}(s) > 0$ (open right-half plane). The corresponding norm is defined as

$$\|F\|_2^2 := \sup_{\sigma>0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(\sigma + j\omega)F(\sigma + j\omega)\right] d\omega\right\}.$$

It can be shown that

$$\|F\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega.$$

The real rational subspace of $\mathcal{H}_2$, which consists of all strictly proper and real rational stable transfer matrices, is denoted by $\mathcal{RH}_2$.

3. $\mathcal{L}_\infty(j\mathbb{R})$ **Space**: $\mathcal{L}_\infty(j\mathbb{R})$ or simply $\mathcal{L}_\infty$ is a Banach space of matrix-valued complex function that are bounded on $j\mathbb{R}$, with norm defined as

$$\|F\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}[F(j\omega)].$$

The real rational subspace of $\mathcal{L}_\infty$, denoted by $\mathcal{RL}_\infty$, consists of all proper and real rational transfer matrices with no poles on imaginary axix.

4. $\mathcal{H}_\infty$ **Space**: $\mathcal{H}_\infty$ is a subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty := \sup_{\text{Re}(s)>0} \sigma_{\max}(F(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(F(j\omega)).$$

The second equality can be regarded as a generalization of the maximum modulus theorem for matrix functions. The real rational subspace of $\mathcal{H}_\infty$ is denoted by $\mathcal{RH}_\infty$, which consists of all proper and real rational stable transfer matrices.

3 Computation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

In this lecture, we mainly consider the norm in the space $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$. Given a stable transfer matrix $\mathbf{T}(s) \in \mathcal{RH}_2$ (or $\mathbf{T}(s) \in \mathcal{RH}_\infty$), the norm $\|\mathbf{T}(s)\|_2$ (or $\|\mathbf{T}(s)\|_\infty$) can, in principle, be computed from its definition, it is useful in many applications to have alternative characterizations and to take advantage of the state space representations of $G(s)$.

Lemma 1. *Consider a transfer matrix*

$$\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

with A stable. Then, we have

$$\|\mathbf{T}\|_{\mathcal{H}_2}^2 = \textit{Trace}(B^\mathsf{T} Q B), \qquad \textit{where } A^\mathsf{T} Q + QA + C^\mathsf{T} C = 0,$$
$$\|\mathbf{T}\|_{\mathcal{H}_2}^2 = \textit{Trace}(CPC^\mathsf{T}), \qquad \textit{where } AP + PA^\mathsf{T} + BB^\mathsf{T} = 0.$$

where Q and P are observability and controllability Gramians.

Interpretation of the $\mathcal{H}_2$ norm of stable transfer matrices:

- **Deterministic interpretation:** Let e_k be the standard unit vector and denote the output

$$\dot{x} = Ax, \quad z = Cx, \quad x(0) = Be_k,$$

by $z_k(t)$. Note that this is the response to an impulse input to the channel k. Since $z_k(t) = Ce^{At}Be_k$, we have

$$\int_0^\infty z_k(t)^\mathsf{T} z_k(t)dt = e_k^\mathsf{T} \left(\int_0^\infty B^\mathsf{T} e^{A^\mathsf{T} t} C^\mathsf{T} C e^{At} B dt \right) e_k.$$

Therefore, Squared $\mathcal{H}_2$ norm is energy sum of transients of output responses:

$$\sum_{k=1}^m \int_0^\infty z_k(t)^\mathsf{T} z_k(t)dt = \int_0^\infty \text{Trace}\left((Ce^{At}B)^\mathsf{T}(Ce^{At}B)\right)dt = \|\mathbf{T}\|_{\mathcal{H}_2}^2.$$

- **Stochastic interpretation:** If w is white noise and $\dot{x} = Ax + Bw, z = Cx$ then

$$\lim_{t\to\infty} \mathbb{E}\left(z(t)^\mathsf{T} z(t)\right) = \|\mathbf{T}\|_{\mathcal{H}_2}^2$$

The squared $\mathcal{H}_2$-norm equals the asymptotic variance of output.

The $\mathcal{H}_2$ of stable transfer matrices can be quantified by the following linear matrix inequalities (LMIs).

Lemma 2. *Consider a transfer matrix*

$$\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

with A stable. Then, we have $\|\mathbf{T}(s)\|_2 < \gamma$ if and only if there exists $P \succ 0$ such that

$$\textit{trace}(CPC^\mathsf{T}) < \gamma^2, \quad \textit{and} \quad AP + PA^\mathsf{T} + BB^\mathsf{T} \prec 0,$$

and there exists $Q \succ 0$ such that

$$\textit{trace}(B^\mathsf{T} Q B) < \gamma^2, \quad \textit{and} \quad A^\mathsf{T} Q + QA + C^\mathsf{T} C \prec 0.$$

Proof. $\Rightarrow$: if $\|\mathbf{T}(s)\|_2 < \gamma$, then we have

$$\text{Trace}(CP_0C^\mathsf{T}) < \gamma^2, \qquad \text{where} \qquad AP_0 + P_0A^\mathsf{T} + BB^\mathsf{T} = 0. \tag{5}$$

Now we consider a perturbed Lyapunov equation

$$AP_\epsilon + P_\epsilon A^\mathsf{T} + BB^\mathsf{T} + \epsilon I = 0,$$

which has a unique solution $P_\epsilon \succ 0, \forall \epsilon > 0$ since A is stable. Note that every element of P_ϵ is a continuous function of $\epsilon > 0$, and

$$\lim_{\epsilon \to 0} P_\epsilon = P_0.$$

Since $\text{Trace}(CP_0C^\mathsf{T}) < \gamma^2$, there exists a $\epsilon > 0$ such that $\text{Trace}(CP_\epsilon C^\mathsf{T}) < \gamma^2$ and

$$AP_\epsilon + P_\epsilon A^\mathsf{T} + BB^\mathsf{T} = -\epsilon I \prec 0.$$

$\Leftarrow$ We need to prove (5). We first have

$$AP + PA^\mathsf{T} + BB^\mathsf{T} - (AP_0 + P_0A^\mathsf{T} + BB^\mathsf{T}) = A(P - P_0) + (P - P_0)A^\mathsf{T} \prec 0.$$

This indicates that $P - P_0 \succ 0$ (since A is stable). Then

$$\text{trace}(CP_0C^\mathsf{T}) < \text{trace}(CPC^\mathsf{T}) < \gamma^2.$$

We have proved that $\|G(s)\|_2 < \gamma$. □

For the computation of $\mathcal{H}_\infty$ norm, we have the following KYP lemma.

Lemma 3. *Consider a transfer matrix*

$$\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with A stable. Then, the following statements are equivalent:

- $\|\mathbf{T}(s)\|_\infty < \gamma$;
- $\mathbf{T}^*(j\omega)\mathbf{T}(j\omega) \prec \gamma^2 I, \forall \omega \in \mathbb{R}$.
- *We have*
$$\sup_{0<\|w\|_2<1} \frac{\|\mathbf{T}(s)w\|_2}{\|w\|_2} < \gamma$$
- *The following LMI is feasible.*
$$\begin{bmatrix} A^\mathsf{T}X + XA & XB & C^\mathsf{T} \\ B^\mathsf{T}X & -\gamma I & D^\mathsf{T} \\ C & D & -\gamma I \end{bmatrix} \prec 0, X \succ 0. \tag{6}$$

We note the LMI (6) has multiple equivalent forms:

$$\begin{bmatrix} A^\mathsf{T}X + XA & XB & C^\mathsf{T} \\ B^\mathsf{T}X & -\gamma^2 I & D^\mathsf{T} \\ C & D & -I \end{bmatrix} \prec 0, X \succ 0.$$

(obtained by left- and right- multiplied by $\text{diag}(\gamma^{\frac{1}{2}}I, \gamma^{\frac{1}{2}}I, \gamma^{-\frac{1}{2}}I)$) and (by applying the Schur complement)

$$\begin{bmatrix} A^\mathsf{T}X + XA + C^\mathsf{T}C & XB + C^\mathsf{T}D \\ B^\mathsf{T}X + D^\mathsf{T}C & D^\mathsf{T}D - \gamma^2 I \end{bmatrix} \prec 0, \qquad X \succ 0,$$

and

$$A^\mathsf{T}X + XA + C^\mathsf{T}C - (XB + C^\mathsf{T}D)(D^\mathsf{T}D - \gamma^2 I)^{-1}(B^\mathsf{T}X + D^\mathsf{T}C) \prec 0, \quad X \succ 0, \;\; \sigma_{\max}(D) < \gamma.$$

which is a Riccati inequality.

4 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control: state feedback

Here, we consider static state feedback $u = D_k x$, and the controller synthesis problem (4) becomes

$$\begin{aligned} \min_{D_k} \quad & \left\| \left[\begin{array}{c|c} A + B_2 D_k & B_1 \\ \hline C_1 + D_{12} D_k & D_{11} \end{array}\right] \right\| \\ \text{subject to} \quad & A + B_2 D_k \text{ is stable.} \end{aligned} \tag{7}$$

4.1 $\mathcal{H}_2$ optimal control

When we aim to minimize the $\mathcal{H}_2$ norm of the closed-loop system $\mathbf{T}_{zw}$, we need to assume $D_{11} = 0$ (otherwise $\mathbf{T}_{zw}$ is not strictly proper and $\|\mathbf{T}_{zw}\|_2$ is not finite). According to Lemma 2, it is easy to know that (7) is equivalent to

$$\begin{aligned} \min_{P, D_k, \gamma} \quad & \gamma \\ \text{subject to} \quad & (A + B_2 D_k) P + P (A + B_2 D_k)^\mathsf{T} + B_1 B_1^\mathsf{T} \prec 0, \\ & \text{trace}((C_1 + D_{12} D_k) P (C_1 + D_{12} D_k)^\mathsf{T}) < \gamma, \\ & P \succ 0. \end{aligned} \tag{8}$$

By introducing $X = D_k P$, this is equivalent to

$$\begin{aligned} \min_{P, X, \gamma} \quad & \gamma \\ \text{subject to} \quad & (AP + B_2 X) + (AP + B_2 X)^\mathsf{T} + B_1 B_1^\mathsf{T} \prec 0, \\ & \text{trace}((C_1 P + D_{12} X) P^{-1} (C_1 P + D_{12} X)^\mathsf{T}) < \gamma, \\ & P \succ 0. \end{aligned} \tag{9}$$

This problem is not convex. Note that

$$\text{trace}((C_1 P + D_{12} X) P^{-1} (C_1 P + D_{12} X)^\mathsf{T}) < \gamma, \quad P \succ 0$$

is equivalent to

$$\begin{bmatrix} Z & C_1 P + D_{12} X \\ (C_1 P + D_{12} X)^\mathsf{T} & P \end{bmatrix} \succ 0, \quad \text{trace}(Z) < \gamma.$$

Therefore, problem (7) is equivalent to

$$\begin{aligned} \min_{P, X, Z} \quad & \text{trace}(Z) \\ \text{subject to} \quad & (AP + B_2 X) + (AP + B_2 X)^\mathsf{T} + B_1 B_1^\mathsf{T} \prec 0, \\ & \begin{bmatrix} Z & C_1 P + D_{12} X \\ (C_1 P + D_{12} X)^\mathsf{T} & P \end{bmatrix} \succ 0, \end{aligned} \tag{10}$$

and the optimal $\mathcal{H}_2$ optimal state feedback gain is recovered by $D_k = X P^{-1}$.

4.2 $\mathcal{H}_\infty$ optimal control

Here, we aim to minimize $\|\mathbf{T}_{zw}\|_\infty$ in (7). According to Lemma 3, problem (7) is equivalent to

$$\begin{aligned} \min_{X, D_k, \gamma} \quad & \gamma \\ \text{subject to} \quad & \begin{bmatrix} (A + B_2 D_k)^{\mathsf{T}} X + X(A + B_2 D_k) & X B_1 & (C_1 + D_{12} D_k)^{\mathsf{T}} \\ B_1^{\mathsf{T}} X & -\gamma I & D_{11}^{\mathsf{T}} \\ C_1 + D_{12} D_k & D_{11} & -\gamma I \end{bmatrix} \prec 0, \\ & X \succ 0. \end{aligned} \tag{11}$$

This is equivalent to

$$\begin{aligned} \min_{P, D_k, \gamma} \quad & \gamma \\ \text{subject to} \quad & \begin{bmatrix} P(A + B_2 D_k)^{\mathsf{T}} + (A + B_2 D_k) P & B_1 & P(C_1 + D_{12} D_k)^{\mathsf{T}} \\ B_1^{\mathsf{T}} & -\gamma I & D_{11}^{\mathsf{T}} \\ (C_1 + D_{12} D_k) P & D_{11} & -\gamma I \end{bmatrix} \prec 0, \\ & P \succ 0, \end{aligned} \tag{12}$$

which is clearly equivalent to

$$\begin{aligned} \min_{P, Y, \gamma} \quad & \gamma \\ \text{subject to} \quad & \begin{bmatrix} (AP + B_2 Y)^{\mathsf{T}} + (AP + B_2 Y) & B_1 & (C_1 + D_{12} Y)^{\mathsf{T}} \\ B_1^{\mathsf{T}} & -\gamma I & D_{11}^{\mathsf{T}} \\ (C_1 + D_{12} Y) & D_{11} & -\gamma I \end{bmatrix} \prec 0, \\ & P \succ 0. \end{aligned} \tag{13}$$

The optimal $\mathcal{H}_\infty$ state feedback gain can be recovered by $D_k = Y P^{-1}$.

5 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control: output feedback

LMI formulations can be derived for the general $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control (4), where the change of variables become much more involved; see [4, Chapter 4] for details. The interested reader is also encouraged to read classical work [2,3]. These problems can be solved via Riccati equations (see [1], and [5, Chpater 14 and Chapter 17]), where some technical conditions are required on the system dynamics. Please refer to [2] for a nice comparsion.

References

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