1. Optimal Control and Semidefinite Optimization: Introduction

Yang Zheng

Postdoc, Harvard University

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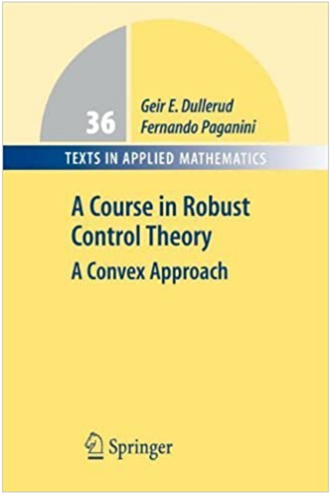
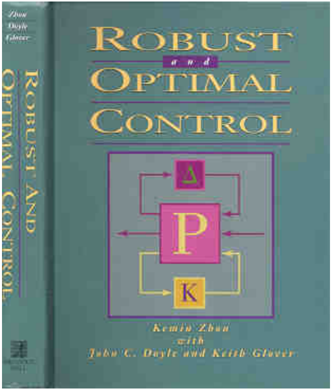
Classical textbooks





Outline

Problem formulation: classical optimal control

Closed-loop systems: frequency and state-space formulation

Basic properties: well-posedness and internal stability

Other topics

Outline

Linear time-invariant systems

- State-space model

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u, \\
z &= C_1 x + D_{11} w + D_{12} u, \\
y &= C_2 x + D_{21} w + D_{22} u,
\end{aligned}
\tag{1}
$$

  where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^d, y \in \mathbb{R}^p, z \in \mathbb{R}^q$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively.

- Frequency-domain model

$$
\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix},
$$

  where $\mathbf{P}_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$. We refer to $\mathbf{P}$ as the open-loop plant model.

LTI dynamic controller

- Frequency-domain model

$$\mathbf{u} = \mathbf{K}\mathbf{y},$$

- State-space realization

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y, \\ u &= C_k \xi + D_k y, \end{aligned} \tag{2}$$

where $\xi \in \mathbb{R}^{n_k}$ is the internal state of controller $\mathbf{K}$. We have

$$\mathbf{K} = C_k (sI - A_k)^{-1} B_k + D_k.$$

- Example: static output feedback controller $\mathbf{K} = D_k$ and observer-based dynamic controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ u &= F\hat{x} \end{aligned} \quad \Big| \quad \begin{aligned} A_k &= A + BF + LC \quad & B_k &= -L \\ C_k &= F & D_k &= 0. \end{aligned}$$

Closed-loop system

- Interconnected system:

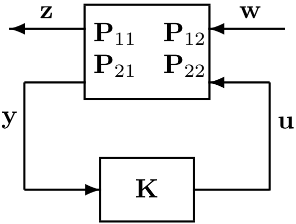


Figure: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

We focus on input-output behaviors. Recall that

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned}$$

- Very general set-up: including LQR/LQG/$\mathcal{H}_2$/$\mathcal{H}_\infty$ control

Optimal control

- Interconnected system:

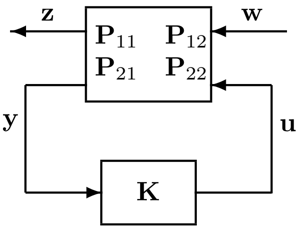


Figure: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

- Informally speaking, we aim to find a controller $\mathbf{K}$ such that the closed-loop system is internally stable and achieves/minimizes desired performance specification:

$$\begin{aligned} \min_{\mathbf{K}} \quad & f(\mathbf{P}, \mathbf{K}) \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{P}. \end{aligned} \tag{3}$$

where $f(\mathbf{P}, \mathbf{K})$ defines a certain performance index.

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Frequency-domain formulation

By (1), we have

$$\mathbf{z} = \mathbf{P}_{11}\mathbf{w} + \mathbf{P}_{12}\mathbf{u},$$
$$\mathbf{y} = \mathbf{P}_{21}\mathbf{w} + \mathbf{P}_{22}\mathbf{u}.$$

Considering the controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, some simple algebra leads to

$$\mathbf{z} = (\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21})\mathbf{w}. \tag{4}$$

Thus, the closed-loop response from $\mathbf{w}$ to $\mathbf{z}$ is

$$\mathbf{T}_{zw} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}.$$

In (3), the cost function is typically chosen as

$$f(\mathbf{P}, \mathbf{K}) = \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\|,$$

where $\|\cdot\|$ can be chosen the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm.

Optimal control

- Optimal control formulation in frequency domain

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{P}. \end{aligned}$$

State-space formulation

- Combining (1) with (2) leads to

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ B_k & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w, \tag{5a}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ 0 & C_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & D_{22} \\ D_k & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} w, \tag{5b}$$

$$z = \begin{bmatrix} C_1 & 0 \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + D_{11} w \tag{5c}$$

- From (5b), we have

$$\begin{bmatrix} I & -D_{22} \\ -D_k & I \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ 0 & C_k \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} D_{21} \\ 0 \end{bmatrix} w, \tag{6}$$

- This equation has a unique solution if and only if the following matrix

$$\begin{bmatrix} I & -D_{22} \\ -D_k & I \end{bmatrix}$$

is invertible, which is equivalent to that $I - D_{22}D_k$ or $I - D_kD_{22}$ is invertible

Well-posedness of feedback systems

Definition

A feedback system is said to be well-posed if the solutions $u(t)$ and $y(t)$ are unique, given any initial condition $x(0)$ and $\xi(0)$ and $w(t)$.

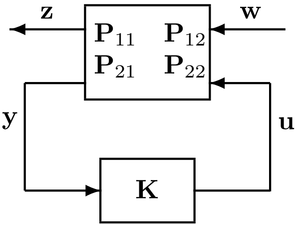

Figure: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

Lemma

The system above is well-posed if and only if $I - D_{22}D_k$ is invertible.

This is equivalent to $I - \mathbf{P}_{22}(\infty)\mathbf{K}(\infty)$ is invertible.

State-space formulation

- It is assumed that the plant is strictly proper, *i.e.* $D_{22} = 0$.
- Now, (6) becomes

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ D_k C_2 & C_k \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} D_{21} \\ D_k D_{21} \end{bmatrix} w$$

Substituting this into (7) leads to

$$\frac{d}{dt} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 + B_2 D_k D_{21} \\ B_k D_{21} \end{bmatrix} w, \qquad \text{(7a)}$$

$$z = \begin{bmatrix} C_1 + D_{12} D_k C_2 & D_{12} C_k \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + (D_{11} + D_{12} D_k D_{21}) w. \qquad \text{(7b)}$$

This is a state-space version of the closed-loop response from w to z. We can write

$$\mathbf{T}_{zw} = \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right].$$

State-space formulation

- Case 1: Static output feedback $\mathbf{K} = D_k$. The closed-loop matrix is $A + B_2 D_k C_2$, and we have

$$\mathbf{T}_{zw} = \left[\begin{array}{c|c} A + B_2 D_k C_2 & B_1 + B_2 D_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{11} + D_{12} D_k D_{21} \end{array}\right].$$

- Case 2, static state feedback $C_2 = I$. The closed-loop matrix is $A + B_2 D_k$

Optimal control

- Optimal control formulation in frequency domain

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{P}. \end{aligned}$$

- Optimal control formulation in state-space domain

$$\begin{aligned} \min_{A_k, B_k, C_k, D_k} \quad & \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{P}. \end{aligned}$$

Outline

Internal stability

Definition

The system in Fig. 1 is *internally stable* if it is well-posed, and the states $(x(t), \xi(t))$ converge to zero as $t \to \infty$ for all initial states $x(0), \xi(0)$ when $w(t) = 0, \forall t$.

Lemma

The system in Fig 3 is internally stable if and only if

$$\hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}$$

is stable.

Internal stability

The set of all stabilizing controllers is defined as

$$\mathcal{C}_{\mathsf{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{P}\}.$$

Then, we have

$$\mathcal{C}_{\mathsf{stab}} = \left\{\mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable}\right\}, \tag{8}$$

where $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. Unfortunately, the stability condition on A_{cl} in (8) is still non-convex in terms of the parameters (A_k, B_k, C_k, D_k).

Summary

- The optimal controller synthesis problem (3) can be precisely written as

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}. \end{aligned}$$

- The state-space version is

$$\begin{aligned} \min_{A_k, B_k, C_k, D_k} \quad & \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\| \\ \text{subject to} \quad & \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.} \end{aligned}$$

- These two formulations are both non-convex in their present forms.

Outline

Problem formulation: classical optimal control

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Other topics

Other topics

- Performance specification: $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of transfer matrices and their computations via convex optimization (LMIs).
- Convex reformulation in the frequency domain (Youla parameterization, system-level synthesis, and closed-loop parameterization).
- Convex reformulation in the state-space domain (convex optimization via LMIs).
- Analytical solutions via solving Algebraic Riccati Equation (ARE).
- Distributed control by introducing a subspace constraint on the controller $\mathbf{K} \in \mathcal{S}$ (Quadratic Invariance, Sparsity Invariance, etc.).