2. Convex reformulation in the frequency domain

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Outline

1. Recap & LQR as as special case
2. External transfer matrix characterization of internal stability
3. Closed-loop parameterization of stabilizing controllers: SLP and IOP
4. Robust stability and it connections with learning-based control
5. Summary

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Linear time-invariant systems

- State-space model

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

  where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^d, y \in \mathbb{R}^p, z \in \mathbb{R}^q$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively.

- Dynamic controller

$$\begin{aligned} \dot{\xi} &= A_k \xi + B_k y, \\ u &= C_k \xi + D_k y, \end{aligned} \tag{2}$$

  where $\xi \in \mathbb{R}^{n_k}$ is the internal state of the controller.

Frequency domain

- Plant model

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix},$$

  where $\mathbf{P}_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$. We refer to $\mathbf{P}$ as the open-loop plant model.
- Controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K} = C_k(sI - A)^{-1}B_k + D_k$.

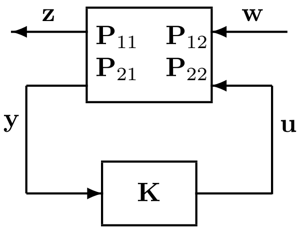

Figure: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

Optimal control

- General optimal control formulation

$$\begin{aligned} \min_{\mathbf{K}} \quad & f(\mathbf{P}, \mathbf{K}) \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{P}. \end{aligned} \tag{3}$$

  where $f(\mathbf{P}, \mathbf{K})$ defines a certain performance index.

- Specifically

Frequency-domain formulation

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{T}_{zw}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}, \end{aligned}$$

where

$$\mathbf{T}_{zw} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}.$$

State-space formulation

$$\min \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\|$$

$$\text{s.t.} \quad \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.}$$

LQR as a special case of $\mathcal{H}_2$ optimal control

Deterministic case

$$\begin{aligned} \min \quad & \int_0^\infty x^\mathsf{T} Q x + u^\mathsf{T} R u \, dt \\ \text{s.t.} \quad & \dot{x} = Ax + Bu \\ & x(0) = x_0, \end{aligned}$$

where $Q \succ 0, R \succ 0$ are weight matrices and $x_0 \in \mathbb{R}^n$.

Stochastic case

$$\begin{aligned} \min \quad & \mathbb{E}\left[\lim_{T\to\infty} \frac{1}{T} \int_0^T x^\mathsf{T} Q x + u^\mathsf{T} R u \, dt\right] \\ \text{s.t.} \quad & \dot{x} = Ax + Bu + w \end{aligned}$$

where $Q \succ 0, R \succ 0$ are weight matrices and $w \sim N(0, I)$.

Both of them are equivalent to

$$\begin{aligned} \min_K \quad & \|\mathbf{T}_{zw}\|_{\mathcal{H}_2}^2 \\ \text{subject to} \quad & \dot{x} = Ax + B_1 w + B_2 u \\ & z = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix} u \\ & u = Kx, \end{aligned}$$

where $B_1 = I, B_2 = B$.

$\mathcal{H}_2$ norm

Consider a stable transfer matrix $\mathbf{T} = C(sI - A)^{-1}B$

$$\|\mathbf{T}\|^2_{\mathcal{H}_2} := \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left(T^*(j\omega)T(j\omega)\right) d\omega$$
$$= \int_0^{\infty} \text{Trace}\left((Ce^{At}B)^\mathsf{T}(Ce^{At}B)\right) dt$$

- **Deterministic interpretation:** Let e_k be the standard unit vector and denote the output

  $$\dot{x} = Ax, \quad z = Cx, \quad x(0) = Be_k,$$

  by $z_k(t)$. Squared $\mathcal{H}_2$ norm is energy sum of output transients:

  $$\sum_{k=1}^{m}\int_0^{\infty} z_k(t)^\mathsf{T} z_k(t)dt = \int_0^{\infty} \text{Trace}\left((Ce^{At}B)^\mathsf{T}(Ce^{At}B)\right) dt = \|\mathbf{T}\|^2_{\mathcal{H}_2}.$$

- **Stochastic interpretation:** If w is white noise and $\dot{x} = Ax + Bw, z = Cx$ then

  $$\lim_{t\to\infty} \mathbb{E}\left(z(t)^\mathsf{T} z(t)\right) = \|\mathbf{T}\|^2_{\mathcal{H}_2}$$

  The squared $\mathcal{H}_2$-norm equals the asymptotic variance of output.

Outline

Static state feedback

- Set of internally stabilizing controllers

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable} \right\},$$

where $\mathbf{K} = C_k(zI - A_k)^{-1} B_k + D_k$.

- Consider a static state feedback case

$$\dot{x} = Ax + B_2 u$$
$$u = D_k x.$$

$$\mathcal{C}_{\text{ss}} = \{D_k \in \mathbb{R}^{m \times n} \mid A + B_2 D_k \text{ is stable}\}.$$

- Lyapunov inequality

$$\begin{aligned} A + B_2 K \text{ is stable} \quad &\iff \quad \exists P \succ 0,\ (A + B_2 K)^{\mathsf{T}} P + P(A + B_2 K) \prec 0 \\ &\iff \quad \exists X \succ 0,\ X(A + B_2 K)^{\mathsf{T}} + (A + B_2 K)X \prec 0 \\ &\iff \quad \exists X \succ 0, Y \in \mathbb{R}^{m \times n},\ XA^{\mathsf{T}} + YB_2^{\mathsf{T}} + AX + B_2 Y \prec 0 \end{aligned}$$

Therefore, we have

$$\mathcal{C}_{\text{ss}} = \{K = YX^{-1} \mid X \succ 0, Y \in \mathbb{R}^{m \times n},\ XA^{\mathsf{T}} + YB_2^{\mathsf{T}} + AX + B_2 Y \prec 0\},$$

External transfer function characterization

Lemma

Consider a transfer matrix $\mathbf{T}(s) = C(sI - A)^{-1}B + D$*. If* (A, B, C) *is detectable and stabilizable, then*

$$\mathbf{T}(s) \in \mathcal{RH}_\infty \quad \Leftrightarrow \quad A \text{ is stable.}$$

Two useful facts:

- The set of stable matrices $\{A \in \mathbb{R}^{n\times n} \mid A \text{ is stable}\}$ is non-convex, but finite-dimensional;
- The set of stable transfer matrices $\{\mathbf{T}(s) \mid \mathbf{T}(s) \in \mathcal{RH}_\infty\}$ is convex, but infinite-dimensional;

External transfer function characterization

Set of internally stabilizing controllers

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable} \right\},$$

where $\mathbf{K} = C_k(zI - A_k)^{-1} B_k + D_k$.

- Consider the plant $\mathbf{P}_{22} = C_2(sI - A)^{-1} B_2$,

$$\dot{x} = Ax + B_2 u + \delta_x,$$
$$y = C_2 x + \delta_y$$

- A dynamic controller

$$\dot{\xi} = A_k \xi + B_k y$$
$$u = C_k \xi + D_k y + \delta_u.$$

Closed-loop responses from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$$

where $\mathbf{Y} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}$, $\mathbf{W} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{22}$, and

$$\mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}, \ \mathbf{Z} = (I - \mathbf{K}\mathbf{P}_{22})^{-1}.$$

External transfer function characterization

Closed-loop responses from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$$

Lemma

The feedback system is internally stable if and only if the transfer matrix from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ is stable.

State-space realization of the transfer matrix from (δ_y, δ_u) to $(\mathbf{y}, \mathbf{u})$ as

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) = \hat{C}_2 (zI - \hat{A})^{-1} \hat{B}_2 + \begin{bmatrix} I & 0 \\ D_k & I \end{bmatrix},$$

where

$$\hat{A} = \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}, \ \hat{B}_2 = \begin{bmatrix} B_2 D_k & B_2 \\ B_k & 0 \end{bmatrix}, \ \hat{C}_2 = \begin{bmatrix} C_2 & 0 \\ D_k C_2 & C_k \end{bmatrix}.$$

It remains to prove that $(\hat{A}, \hat{B}_2)$ is stabilizable and $(\hat{A}, \hat{C}_2)$ is detectable.

External transfer function characterization

The stabilizability of $(\hat{A}, \hat{B}_2)$ can be seen from the following fact

$$\begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} + \begin{bmatrix} B_2 D_k & B_2 \\ B_k & 0 \end{bmatrix} \begin{bmatrix} -C_2 & F_k \\ F & 0 \end{bmatrix} = \begin{bmatrix} A + B_2 F & B_2 C_k + B_2 D_k F_k \\ 0 & A_k + B_k F_k, \end{bmatrix}$$

which will be stable if $A + B_2 F$ and $A_k + B_k F_k$ are stable. The detectability of $(\hat{A}, \hat{C}_2)$ can be shown in a similar way.

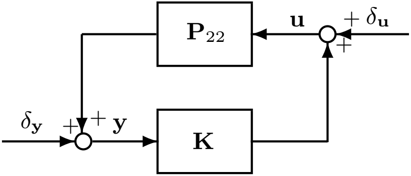

Figure: Input-output stability

External transfer matrix characterization

Look at closed-loop response from (δ_x, δ_y) to $\mathbf{x}, \mathbf{u}$. It is not difficult to derive that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix},$$

where $\mathbf{R} = (zI - A - B_2\mathbf{K}C_2)^{-1}, \mathbf{M} = \mathbf{K}C_2\mathbf{R}$, and

$$\mathbf{U} = \mathbf{R}B_2\mathbf{K}, \quad \mathbf{L} = \mathbf{K}C_2\mathbf{R}B_2\mathbf{K} + \mathbf{K}.$$

Lemma

The feedback system is internally stable if and only if the transfer matrix from (δ_x, δ_y) to $(\mathbf{x}, \mathbf{u})$ is stable.

A state-space realization of the transfer matrix from (δ_x, δ_y) to $(\mathbf{x}, \mathbf{u})$ as

$$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \to \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) = \hat{C}_1 (zI - \hat{A})^{-1} \hat{B}_1 + \begin{bmatrix} 0 & 0 \\ 0 & D_k \end{bmatrix},$$

where

$$\hat{B}_1 = \begin{bmatrix} I & B_2 D_k \\ 0 & B_k \end{bmatrix}, \quad \hat{C}_1 = \begin{bmatrix} I & 0 \\ D_k C_2 & C_k \end{bmatrix}$$

A summary

Four equivalent statements

- $\mathbf{K}$ internally stabilizes the plant $\mathbf{P}$;
- $\hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}$ is stable
- The following closed-loop responses are stable (input-output parameterization)

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$$

- The following closed-loop responses are stable (system-level parameterization)

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix},$$

Two special cases

- **Open-loop stable plants:**

Corollary

If the LTI system is open-loop stable (i.e., A is stable), then $\mathbf{K} \in \mathcal{C}_{stab}$ if and only if $(\delta_y \to \mathbf{u}) := \mathbf{U} \in \mathcal{RH}_\infty$.

The state-space representation is

$$\mathbf{U} = \begin{bmatrix} D_k C & C_k \end{bmatrix} (zI - \hat{A})^{-1} \begin{bmatrix} BD_k \\ B_k \end{bmatrix} + D_k.$$

Considering the fact that the following matrix

$$\hat{A} + \begin{bmatrix} BD_k \\ B_k \end{bmatrix} \begin{bmatrix} -C & F_k \end{bmatrix} = \begin{bmatrix} A & BC_k + BD_k F_k \\ 0 & A_k + B_k F_k \end{bmatrix},$$

is stable when A and $A_k + B_k F_k$ are stable.

- **State feedback**

Corollary

If $C = I$, then $\mathbf{K} \in \mathcal{C}_{stab}$ if and only if $\left(\delta_x \to \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}\right) := \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \in \mathcal{RH}_\infty$.

Outline

Closed-loop parameterization: IOP

Input-output parameterization

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$$

Corollary

If the LTI system is open-loop stable, then we have

$$\mathcal{C}_{stab} = \left\{ \mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \middle| \begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{U} \end{bmatrix} = I, \ \mathbf{U} \in \mathcal{RH}_\infty \right\}.$$

$\Rightarrow$ With $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, it is not difficult to derive

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{P}_{22}\mathbf{K})^{-1} \\ \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} \end{bmatrix} \delta_y.$$

Let us define $\mathbf{Y} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}$ and $\mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}$.
Since $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, we know that $\mathbf{U} \in \mathcal{RH}_\infty$. Also, by definition, $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1}$.
Finally, it is very easy to verify that

$$\mathbf{Y} - \mathbf{P}_{22}\mathbf{U} = (I - \mathbf{P}_{22}\mathbf{K})^{-1} - \mathbf{P}_{22}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} = I.$$

Closed-loop parameterization: IOP

Input-output parameterization

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$$

Corollary

If the LTI system is open-loop stable, then we have

$$\mathcal{C}_{stab} = \left\{ \mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \middle| \begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{U} \end{bmatrix} = I,\ \mathbf{U} \in \mathcal{RH}_\infty \right\}.$$

$\Leftarrow$. Given $\mathbf{Y}$ and $\mathbf{U}$ satisfying the condition, we show that $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{C}_{\text{stab}}$.
We only need to show the response from δ_y to $\mathbf{u}$ is Stable.
In particular, we have

$$\begin{aligned} \mathbf{u} &= \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\delta_y \\ &= \mathbf{U}\mathbf{Y}^{-1}(I - \mathbf{P}_{22}\mathbf{U}\mathbf{Y}^{-1})^{-1}\delta_y \\ &= \mathbf{U}\delta_y, \end{aligned}$$

where the last equality used the affine relationship $\mathbf{Y} - \mathbf{P}_{22}\mathbf{U} = I$.

Closed-loop parameterization: SLP

System-level parameterization

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{R} & \mathbf{N}\\ \mathbf{M} & \mathbf{L}\end{bmatrix}\begin{bmatrix}\delta_x\\ \delta_y\end{bmatrix},$$

Corollary

If $C_2 = I$, then we have

$$\mathcal{C}_{stab} = \left\{\mathbf{K} = \mathbf{M}\mathbf{R}^{-1} \,\middle|\, \begin{bmatrix}zI - A & -B_2\end{bmatrix}\begin{bmatrix}\mathbf{R}\\ \mathbf{M}\end{bmatrix} = I,\ \mathbf{M}, \mathbf{R} \in \mathcal{RH}_\infty\right\}.$$

$\Leftarrow$ Consider $\mathbf{M}, \mathbf{R}$ satisfying the condition. We define $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$, and show this controller $\mathbf{K} \in \mathcal{C}_{\text{stab}}$.
It is sufficient to show

$$\mathbf{x} = (sI - A - B_2\mathbf{K})^{-1}\delta_x = (sI - A - B_2\mathbf{M}\mathbf{R}^{-1})^{-1}\delta_x = \mathbf{R}\delta_x$$
$$\mathbf{u} = \mathbf{K}(sI - A - B_2\mathbf{K})^{-1}\delta_x = \mathbf{M}\delta_x$$

General case: Input-output parameterization

$$\begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{4a}$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{P}_{22} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{4b}$$

$$\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \in \mathcal{RH}_\infty. \tag{4c}$$

Theorem (Input-output parameterization)

The set of all internally stabilizing controllers can be represented as

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \text{ are in the affine subspace (4a)-(4c)}\}.$$

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}, \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (4a) - (4c). \end{aligned}$$

General case: System-level synthesis

$$\begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{5a}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} sI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{5b}$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty. \tag{5c}$$

Theorem (System-level parameterization)

For strictly proper plants, the set of all internally stabilizing controllers can be represented as

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R},\ \mathbf{M},\ \mathbf{N},\ \mathbf{L} \textit{ are in the affine subspace } \text{(5a)-(5c)}\}.$$

System-level synthesis

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & \text{(5a)} - \text{(5c)}. \end{aligned}$$

Summary

- Optimal controller synthesis problem

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}. \end{aligned}$$

Four equivalent statements

- $\mathbf{K}$ internally stabilizes the plant $\mathbf{P}$;
- $\hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}$ is stable
- The following closed-loop responses are stable (IOP)
$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$
- The following closed-loop responses are stable (SLP)
$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix},$

Other issues

- Numerical computation: Finite impulse responses (only work for discrete-time systems)

$$\mathbf{H}(z) = \sum_{k=1}^{T} H_k \frac{1}{z^k}$$

- See a Github repository here: `https://github.com/zhengy09/h2_clp`.
- Distributed control $\mathbf{K} \in \mathcal{S}$:

$$\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{S}$$
$$\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \in \mathcal{S}$$

- State-space realization of these controllers
- Numerical robustness: the affine constraints can never be exactly satisfied in numerical computaiton ...

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Robust stability: SLP in the state feedback case

- Suppose we only have estimation $\hat{A}$ and $\hat{B}_2$, where $\|A - \hat{A}\| \leq \epsilon_A$ and $\|B - \hat{B}_2\| \leq \epsilon_B$.
- How can we design a stabilizing controller for the true system (A, B_2) based on the information $(\hat{A}, \hat{B}_2)$ and ϵ_A, ϵ_B?
- We find $\hat{\mathbf{M}}, \hat{\mathbf{R}} \in \mathcal{RH}_\infty$ that satisfies

$$\begin{bmatrix} sI - \hat{A} & -\hat{B}_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix} = I. \tag{6}$$

- Then, the controller $\mathbf{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ stabilizes $(\hat{A}, \hat{B}_2)$. What happens if we apply $\mathbf{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ to the true system (A, B_2)?
- we have

$$\begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \end{bmatrix} = I + \mathbf{\Delta},$$

  where $\mathbf{\Delta} = \Delta_A \hat{\mathbf{R}} + \Delta_B \hat{\mathbf{M}}$. Then it is not difficult to show that if $\|\Delta\|_\infty < 1$, the controller $\mathbf{K} = \hat{\mathbf{M}}\hat{\mathbf{R}}^{-1}$ stabilizes the true system (A, B_2) as well.

Robust stability: IOP for open-loop stable plants

- Suppose we have the transfer matrix estimation $\hat{\mathbf{P}}_{22}$, with $\|\mathbf{P}_{22} - \hat{\mathbf{P}}_{22}\|_\infty \leq \epsilon$.
- We find $\hat{\mathbf{Y}}, \hat{\mathbf{U}} \in \mathcal{RH}_\infty$ that satisfies

$$\begin{bmatrix} I & -\hat{\mathbf{P}}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} \\ \hat{\mathbf{U}} \end{bmatrix} = I.$$

- Then, the controller $\mathbf{K} = \hat{\mathbf{U}}\hat{\mathbf{Y}}^{-1}$ stabilizes the plant $\hat{\mathbf{P}}_{22}$.
- For the true plant $\mathbf{P}_{22}$, we have

$$\begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} \\ \hat{\mathbf{U}} \end{bmatrix} = I + \mathbf{\Delta},$$

  where $\mathbf{\Delta} = \Delta \mathbf{P}_{22}$.
- Then it is not difficult to show that if $\|\Delta\|_\infty < 1$, the controller $\mathbf{K} = \hat{\mathbf{U}}\hat{\mathbf{Y}}^{-1}$ stabilizes the true system $\mathbf{P}_{22}$ as well.

Outline

A summary

Four equivalent statements

- $\mathbf{K}$ internally stabilizes the plant $\mathbf{P}$;
- $\hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix}$ is stable
- The following closed-loop responses are stable (input-output parameterization)
$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix},$$
- The following closed-loop responses are stable (system-level parameterization)
$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix},$$

Summary

- Optional controller systhesis

$$\begin{aligned}\min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}},\end{aligned}$$

- Input-output parameterization

$$\begin{aligned}\min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \text{(4a)} - \text{(4c)}.\end{aligned}$$

- System-level parameterization

$$\begin{aligned}\min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & \text{(5a)} - \text{(5c)}.\end{aligned}$$

Other topics

- State-space formulation

$$
\min \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array}\right] \right\|
$$

$$
\text{s.t.} \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.}
$$

- How to derive convex reformulation in the state space? This needs to specify the norm of the cost function. We can have LMI formulations (the topic in the next week).

- How to deal with structural constraint $\mathcal{K} \in \mathcal{S}$?