3. Youla Parameterization and Disturbance Feedback Implementation

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Outline

1. Youla for open-loop stable plants
2. Youla in finite-time horizon and disturbance feedback
3. Doubly co-prime factorization and Youla for general plants
4. Equivalence with SLP and IOP

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Internally stabilizing controllers

- Consider a linear time-invariant system

$$\dot{x} = Ax + B_2 u + \delta_x,$$
$$y = C_2 x + \delta_y,$$

- A dynamic output feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K}$ has a state-space realization

$$\dot{\xi} = A_k \xi + B_k y,$$
$$u = C_k \xi + D_k y,$$

  with $\xi \in \mathbb{R}^{n_k}$ being the internal state of controller $\mathbf{K}$.

- Define the set of internally stabilizing controllers as

$$\mathcal{C}_{\text{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{P}\},$$

- A state-space characterization is

$$\mathcal{C}_{\text{stab}} = \left\{\mathbf{K} \mid \hat{A} := \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable}\right\},$$

  where $\mathbf{K} = C_k(zI - A_k)^{-1} B_k + D_k$.

Youla for open-loop stable plants

Theorem

Suppose the plant is open-loop stable, i.e. A is stable. Then, the set of all stabilizing controllers can be represented as

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\},$$

where $\mathbf{G} = C_2(sI - A)^{-1}B_2$.

$\Rightarrow$: Suppose $\mathbf{K}_0 \in \mathcal{C}_{\text{stab}}$. Then, we have $\mathbf{Q}_0 := \mathbf{K}_0(I - \mathbf{G}\mathbf{K}_0)^{-1} \in \mathcal{RH}_\infty$. It can be verified that $\mathbf{K}_0$ can be expressed as follows

$$\mathbf{Q}_0(I + \mathbf{G}\mathbf{Q}_0)^{-1} = \mathbf{K}_0(I - \mathbf{G}\mathbf{K}_0)^{-1}(I + \mathbf{G}\mathbf{K}_0(I - \mathbf{G}\mathbf{K}_0)^{-1})^{-1} = \mathbf{K}_0.$$

$\Leftarrow$: Suppose $\mathbf{Q} \in \mathcal{RH}_\infty$, and define $\mathbf{K} = \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1}$. Since the plant is open-loop stable, we only need to check the closed-loop response from δ_y to $\mathbf{u}$ is stable.

$$\begin{aligned}\mathbf{u} &= \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\delta_y \\ &= \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1}(I - \mathbf{G}\mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1})^{-1}\delta_y \\ &= \mathbf{Q}\delta_y.\end{aligned}$$

Disturbance feedback implementation

The controller $\mathbf{K} = \mathbf{Q}(I + \mathbf{GQ})^{-1}$ can be implemented in a disturbance-based form:

$$\beta = \mathbf{y} - \mathbf{Gu},$$
$$\mathbf{u} = \mathbf{Q}\beta.$$

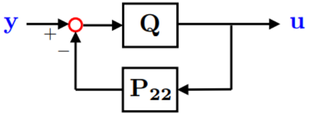

Figure: Internal model principle, where $\mathbf{P}_{22} := \mathbf{G}$.

Disturbance feedback implementation

The controller $\mathbf{K} = \mathbf{Q}(I + \mathbf{G}\mathbf{Q})^{-1}$ can be implemented in a disturbance-based form:

$$\beta = \mathbf{y} - \mathbf{G}\mathbf{u},$$
$$\mathbf{u} = \mathbf{Q}\beta.$$

Truncation in Finite Impulse Response form: Consider the following forms

$$\mathbf{G} = \sum_{k=1}^{p} G_k \frac{1}{z^k}, \qquad \mathbf{Q} = \sum_{k=0}^{q} Q_k \frac{1}{z^k},$$

then the controller can be implemented as

$$\beta_t = y_t - \sum_{k=1}^{p} G_k u_{t-k},$$
$$u_t = \sum_{k=0}^{q} Q_k \beta_{t-k}.$$

Internal Model Principle

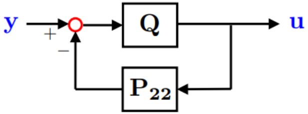

Figure: Internal model principle, where $\mathbf{P}_{22} := \mathbf{G}$.

- Known as the internal model principle [1], applied in Youla.
- The following paragraph is quoted from [2]:
  "*The concept of internal models plays a crucial role in regulator problems. The internal model principle can intuitively be expressed as: 'Any good regulator must create a model of the dynamic structure of the environment in the closed loop system'*".

1. Bruce A Francis and Walter Murray Wonham. The internal model principle of control theory.Automatica, 12(5):457–465, 1976
2. Gunnar Bengtsson. Output regulation and internal models—a frequency domain approach.Automatica, 13(4):333–345, 1977.

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Discrete-time LTI systems

- Consider a discrete-time system

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^n$ is the disturbance.

- Mixed constraints on the state and input:

$$\mathcal{Z} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Cx + Du \leq b\},$$

where the matrices $C \in \mathbb{R}^{s\times n}, D \in \mathbb{R}^{s\times m}$ and the vector $b \in \mathbb{R}^s$. It is assumed that Z is bounded and contains the origin in its interior.

- A target/terminal constraint set X_f is given by

$$X_f := \{x \in \mathbb{R}^n \mid Yx \leq z\}, \tag{1}$$

where the matrix $Y \in \mathbb{R}^{r\times n}$ and the vector $z \in \mathbb{R}^r$. It is assumed that X_f is bounded and contains the origin in its interior.

Goal: Find a control sequence u_t that satisfies the above requirements.

Plan for a finite horizon

- Predictions of the system's evolution over a finite control/planning horizon:
$$\mathbf{x} := \left[x_0^\top, \ldots, x_N^\top\right]^\top \in \mathbb{R}^{n(N+1)},$$
$$\mathbf{u} := \left[u_0^\top, \ldots, u_{N-1}^\top\right]^\top \in \mathbb{R}^{mN},$$
$$\mathbf{w} := \left[w_0^\top, \ldots, w_{N-1}^\top\right]^\top \in \mathbb{R}^{nN},$$
where $x_0 = x$ denotes the current measured value of the state.
- Let the set $\mathcal{W} := W \times \ldots \times W$, so that $\mathbf{w} \in \mathcal{W}$.
- The system can be compactly written as
$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w},$$
where
$$\mathbf{A} = \begin{bmatrix} I & 0 & 0 & \ldots & 0 \\ A & 0 & 0 & \ldots & 0 \\ 0 & A & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ B & 0 & \ldots & 0 \\ 0 & B & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & B \end{bmatrix}, \mathbf{E} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ I & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & I \end{bmatrix}.$$

State feedback policies

- Search over the set of time-varying affine state feedback control policies with knowledge of prior states:

$$u_t = \sum_{i=0}^{t} L_{t,i} x_i + g_t, \qquad t = 0, \ldots, N-1,$$

  where each $L_{t,i} \in \mathbb{R}^{m \times n}$ and $g_t \in \mathbb{R}^m$.

- Define the block lower triangular matrix $\mathbf{L} \in \mathbb{R}^{mN \times n(N+1)}$ and stacked vector $\mathbf{g} \in \mathbb{R}^{mN}$ as

$$\mathbf{L} = \begin{bmatrix} L_{0,0} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N_1,0} & \ldots & L_{N_1,N-1} & 0 \end{bmatrix}, \mathbf{g} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}. \qquad (2)$$

  Then, the input sequence can be written as

$$\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}.$$

Feasibility and convexity of state-feedback policies

The set of admissible $(\mathbf{L}, \mathbf{g})$ is defined as

$$\Pi_N^{\mathsf{sf}}(x) := \left\{ (\mathbf{L}, \mathbf{g}) \;\middle|\; \begin{array}{l} (\mathbf{L}, \mathbf{g}) \text{ satisfy (2)}, x_0 = x, \\ x_{t+1} = Ax_t + Bu_t + w_t \\ u_t = \sum_{i=0}^{t} L_{t,i} x_i + g_t \\ (x_t, u_t) \in Z, x_N \in X_f \\ t = 0, \dots, N-1, \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}.$$

Proposition

The set of admissible affine state feedback parameters $\Pi_N^{sf}(x)$ is non-convex.

Disturbance feedback policies

- Parameterize the control policy as an affine function of the sequence of past disturbances, so that

$$u_t = \sum_{i=0}^{t-1} M_{t,i} w_t + v_t, \qquad t = 0, \dots, N-1$$

where each $M_{t,i} \in \mathbb{R}^{m \times n}$ and $v_t \in \mathbb{R}^m$.

- The past disturbance sequence is easily calculated as

$$w_{t-1} = x_t - Ax_{t-1} - Bu_{t-1}.$$

- We define the vector $\mathbf{v} \in \mathbf{R}^{mN}$ and the strictly block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ M_{1,0} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \dots & M_{N-1,N-2} & 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}. \qquad (3)$$

- Then, the input sequence can be written as

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}.$$

Feasibility and convexity of disturbance-feedback policies

The set of admissible $(\mathbf{M}, \mathbf{v})$ is defined as

$$\Pi_N^{\text{df}}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfy (3)}, x_0 = x, \\ x_{t+1} = Ax_t + Bu_t + w_t \\ u_t = \sum_{i=0}^{t-1} M_{t,i} w_i + v_t \\ (x_t, u_t) \in Z, x_N \in X_f \\ t = 0, \ldots, N-1, \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}.$$

Proposition

The set of admissible affine state feedback parameters $\Pi_N^{df}(x)$ is convex.

Feasibility and convexity of disturbance feedback policies

- The state and input sequences can be written as

$$\mathbf{x} = (I - \mathbf{A})^{-1}(\mathbf{BM} + \mathbf{E})\mathbf{w} + (I - \mathbf{A})^{-1}\mathbf{Bv},$$
$$\mathbf{u} = \mathbf{Mw} + \mathbf{v}.$$

- We have

$$\Pi_N^{\mathsf{df}}(x) := \left\{ (\mathbf{M}, \mathbf{v}) \;\middle|\; \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfy (3)} \\ F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \leq c + Hx, \\ \forall \mathbf{w} \in \mathcal{W} \end{array} \right\}.$$

with some matrices F, G, H, c.

Equivalence

Theorem

For any admissible $(\mathbf{L}, \mathbf{g})$, an admissible $(\mathbf{M}, \mathbf{v})$ can be found that yields the same state and input sequence for all allowable disturbance sequences, and vice-versa.

$\Rightarrow$ Given $(\mathbf{L}, \mathbf{g})$, we find $(\mathbf{M}, \mathbf{v})$ that yields the same state and input sequence. First, we have

$$\begin{aligned} \mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{L}\mathbf{x} + \mathbf{g}) + \mathbf{E}\mathbf{w} \\ \Rightarrow \mathbf{x} &= (I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w} + (I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{B}\mathbf{g} \\ \Rightarrow \mathbf{u} &= \mathbf{L}(I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}\mathbf{w} + \mathbf{L}(I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{B}\mathbf{g} + \mathbf{g} \end{aligned}$$

Let us define

$$\mathbf{M} = \mathbf{L}(I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}, \qquad \mathbf{v} = \mathbf{L}(I - \mathbf{A} - \mathbf{B}\mathbf{L})^{-1}\mathbf{B}\mathbf{g} + \mathbf{g},$$

then, the closed-loop system with $(\mathbf{M}, \mathbf{v})$ yields the same state and input sequence.

It is routinely to show that $(\mathbf{M}, \mathbf{v})$ has the same structure in (3).

$\Leftarrow$: Almost similar; see [Goulart et al., 2006, Automatica] for details.

A summary

- State-feedback policies

$$u_t = \sum_{i=0}^{t} L_{t,i} x_i + g_t, \qquad t = 0, \ldots, N-1,$$

  where each $L_{t,i} \in \mathbb{R}^{m \times n}$ and $g_t \in \mathbb{R}^m$.

- Disturbance feedback polices

$$u_t = \sum_{i=0}^{t-1} M_{t,i} w_t + v_t, \qquad t = 0, \ldots, N-1$$

  where each $M_{t,i} \in \mathbb{R}^{m \times n}$ and $v_t \in \mathbb{R}^m$.

They are equivalent to each other.

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Doubly co-prime factorization

A collection of stable transfer functions, $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r$ is called a doubly co-prime factorization of $\mathbf{G}$ if

$$\mathbf{G} = \mathbf{N}_r \mathbf{M}_r^{-1} = \mathbf{M}_l^{-1} \mathbf{N}_l$$

and

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = I.$$

- We have the following equivalence

  $$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\},$$

  where $\mathbf{Q}$ is denoted as the Youla parameter.
- For open-loop stable system $\mathbf{G}$, we can choose

  $$\mathbf{U}_l = I, \mathbf{V}_l = 0, \mathbf{N}_l = \mathbf{G}, \mathbf{M}_l = I,$$
  $$\mathbf{U}_r = I, \mathbf{V}_r = 0, \mathbf{N}_r = \mathbf{G}, \mathbf{M}_r = I.$$

Optimal controller synthesis

Classical Optimal control

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{GK})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{G}. \end{aligned}$$

Convex reformulation in Youla

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty, \end{aligned}$$

where $\mathbf{T}_{11} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{V}_r\mathbf{M}_l\mathbf{P}_{21}, \mathbf{T}_{12} = -\mathbf{P}_{12}\mathbf{M}_r$, and $\mathbf{T}_{21} = \mathbf{M}_l\mathbf{P}_{21}$.

- It is *an equivalent change of variables*

$$\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}$$

that allows for convexification.

Computation of doubly co-prime factorization

Theorem

Suppose $\mathbf{G}(s)$ *is a proper real-rational matrix and*

$$\mathbf{G} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

is a stabilizable and detectable realization. Let F *and* L *be such that* $A + BF$ *and* $A + LC$ *are both stable, and a doubly co-prime factorization of* $\mathbf{G}$ *is as follows.*

$$\begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B & L \\ \hline F & I & 0 \\ C + DF & D & I \end{array}\right],$$

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -(B + LD) & -L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right],$$

Feedback interpretation

- Consider the state-space model

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

- Next, introduce a state feedback and change the variable

$$v := u - Fx$$

where F is such that $A + BF$ is stable.

- Then, we get

$$\dot{x} = (A + BF)x + Bv,$$
$$u = Fx + v$$
$$y = (C + DF)x + Dv.$$

- From these equations, the transfer matrix from v to u and from v to y are

$$\mathbf{M}_r(s) = \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \end{array}\right], \qquad \mathbf{N}_r(s) = \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & d \end{array}\right].$$

- Therefore, we have $\mathbf{u} = \mathbf{M}_r\mathbf{v},\ \mathbf{y} = \mathbf{N}_r\mathbf{v}$, so that $\mathbf{y} = \mathbf{N}_r\mathbf{M}_r^{-1}\mathbf{u}$, *i.e.*, $\mathbf{G} = \mathbf{N}_r\mathbf{M}_r^{-1}$.

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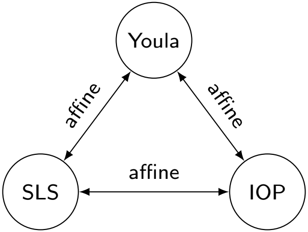
Explicit equivalence among Youla, SLS, and IOP

— any convex SLS can be equivalently reformulated into a convex problem in Youla or IOP; vice versa



Youla $\Leftrightarrow$ IOP

Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{G}$. We have

1. For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\begin{aligned}\mathbf{Y} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l\,,\\ \mathbf{U} &= (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l\,,\\ \mathbf{W} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{N}_l\,,\\ \mathbf{Z} &= I + (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{N}_l\,,\end{aligned}$$

   belong to the IOP constraint and are such that

$$\mathbf{U}\mathbf{Y}^{-1} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}.$$

2. For any $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$ in the IOP constraint, the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l\mathbf{Y}\mathbf{U}_r - \mathbf{U}_l\mathbf{U}\mathbf{U}_r - \mathbf{V}_l\mathbf{W}\mathbf{V}_r + \mathbf{U}_l\mathbf{Z}\mathbf{V}_r - \mathbf{V}_l\mathbf{U}_r\,,$$

   is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} = \mathbf{U}\mathbf{Y}^{-1}$.

IOP $\Leftrightarrow$ SLS

For any $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ satisfying the SLP constraint, the transfer matrices

$$\begin{aligned}
\mathbf{Y} &= C_2\mathbf{N} + I, \\
\mathbf{U} &= \mathbf{L}, \\
\mathbf{W} &= C_2\mathbf{R}B_2, \\
\mathbf{Z} &= \mathbf{M}B_2 + I,
\end{aligned}$$

belong to the IOP constraint and are such that

$$\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} = \mathbf{U}\mathbf{Y}^{-1}.$$

- The affine relationship can written into

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} C_2 & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} & B_2 \\ I & \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

- *This affine transformation is in general not invertible, but considering the achievability conditions, an explicit converse transformation can be found as well.*

IOP $\Leftrightarrow$ SLS

For any $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ satisfying the IOP constraint, the transfer matrices

$$\begin{aligned}
\mathbf{R} &= (sI - A)^{-1} + (sI - A)^{-1} B_2 \mathbf{U} C_2 (sI - A)^{-1} \\
\mathbf{M} &= \mathbf{U} C_2 (sI - A)^{-1}, \\
\mathbf{N} &= (sI - A)^{-1} B_2 \mathbf{U}, \\
\mathbf{L} &= \mathbf{U},
\end{aligned}$$

belong to the SLP constraint and are such that

$$\mathbf{U}\mathbf{Y}^{-1} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}.$$

Youla $\Leftrightarrow$ SLS

Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{G}$. We have

1. For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\begin{aligned} \mathbf{R} &= (sI - A)^{-1} + (sI - A)^{-1} B_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l C_2 (sI - A)^{-1} \\ \mathbf{M} &= (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l C_2 (sI - A)^{-1}, \\ \mathbf{N} &= (sI - A)^{-1} B_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \\ \mathbf{L} &= (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \end{aligned}$$

   belong to the SLP constraint and are such that

$$\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1}.$$

2. For any $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ in the SLP constraint, the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l C_2 \mathbf{N} \mathbf{U}_r - \mathbf{U}_l \mathbf{L} \mathbf{U}_r - \mathbf{V}_l C_2 \mathbf{R} B_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{M} B_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{V}_r$$

   is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$.

Youla ⇔ SLS ⇔ IOP

Convex system-level synthesis: (Wang et al., 2019)

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & g(\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}) \\ \text{subject to} \quad & \text{SLP constraint}, \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

- This is clearly equivalent to a convex problem in Youla,

$$\begin{aligned} \min_{\mathbf{Q}} \quad & g_1(\mathbf{Q}) \\ \text{subject to} \quad & \begin{bmatrix} f_1(\mathbf{Q}) & f_3(\mathbf{Q}) \\ f_2(\mathbf{Q}) & f_4(\mathbf{Q}) \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

- which is also equivalent to a convex problem in input-output parameterization

$$\begin{aligned} \min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \hat{g}_1(\mathbf{U}) \\ \text{subject to} \quad & \text{IOP constraint} \\ & \begin{bmatrix} \hat{f}_1(\mathbf{U}) & \hat{f}_3(\mathbf{U}) \\ \hat{f}_2(\mathbf{U}) & \hat{f}_4(\mathbf{U}) \end{bmatrix} \in \mathcal{S}. \end{aligned}$$