4. LMI formulations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control

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Outline

1. Recap: optimal control and its convex formulations
2. Hardy spaces: $\mathcal{H}_2$ and $\mathcal{H}_\infty$, and their LMI computations
3. LMI formulation for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control: state-feedback
4. Distributed control and Quadratic Invariance

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Linear time-invariant systems

- State-space model

$$\dot{x} = Ax + B_1 w + B_2 u,$$
$$z = C_1 x + D_{11} w + D_{12} u,$$
$$y = C_2 x + D_{21} w + D_{22} u,$$

  where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^d, y \in \mathbb{R}^p, z \in \mathbb{R}^q$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively.

- Dynamic controller

$$\dot{\xi} = A_k \xi + B_k y,$$
$$u = C_k \xi + D_k y,$$

  where $\xi \in \mathbb{R}^{n_k}$ is the internal state of the controller.

Frequency domain

- Plant model

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix},$$

  where $\mathbf{P}_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$. We refer to $\mathbf{P}$ as the open-loop plant model.
- Controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, where $\mathbf{K} = C_k(sI - A)^{-1}B_k + D_k$.

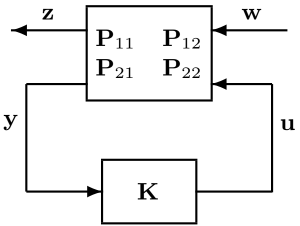


Figure: Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$

Optimal control

- General optimal control formulation

$$\min_{\mathbf{K}} \quad f(\mathbf{P}, \mathbf{K})$$
$$\text{subject to} \quad \mathbf{K} \text{ internally stabilizes } \mathbf{P}.$$

  where $f(\mathbf{P}, \mathbf{K})$ defines a certain performance index.

- Specifically

Frequency-domain formulation

$$\min_{\mathbf{K}} \quad \|\mathbf{T}_{zw}\|$$
$$\text{subject to} \quad \mathbf{K} \in \mathcal{C}_{\text{stab}},$$

where

$$\mathbf{T}_{zw} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}.$$

State-space formulation

$$\min \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\|$$

$$\text{s.t.} \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.}$$

Input-output parameterization

Consider the closed-loop responses from (δ_y, δ_u) to (y, u):

$$\begin{bmatrix} I & -\mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{1a}$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{P}_{22} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{1b}$$

$$\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \in \mathcal{RH}_\infty. \tag{1c}$$

Theorem (Input-output parameterization)

The set of all internally stabilizing controllers can be represented as

$$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \text{ are in the affine subspace (1a)-(1c)}\}.$$

$$\min_{\mathbf{K}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\|$$
subject to $\mathbf{K} \in \mathcal{C}_{\text{stab}}$,

$$\min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\|$$
subject to (1a) − (1c).

System-level parameterization

Consider the closed-loop responses from $(\boldsymbol{\delta}_x, \boldsymbol{\delta}_y)$ to $(\boldsymbol{x}, \boldsymbol{u})$:

$$\begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{2a}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} sI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{2b}$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty. \tag{2c}$$

Theorem (System-level parameterization)

For strictly proper plants, the set of all internally stabilizing controllers can be represented as

$\mathcal{C}_{stab} = \{\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R},\ \mathbf{M},\ \mathbf{N},\ \mathbf{L}$ *are in the affine subspace* (2a)-(2c)$\}$.

System-level synthesis

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & \text{(2a)} - \text{(2c)}. \end{aligned}$$

Youla parameterization

- **Classical Optimal control**

$$\begin{aligned} \min_{\mathbf{K}} \quad & \left\|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{P}_{21}\right\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{G}. \end{aligned}$$

- We have the following equivalence

$$\mathcal{C}_{\text{stab}} = \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\},$$

  where $\mathbf{Q}$ is denoted as the Youla parameter.

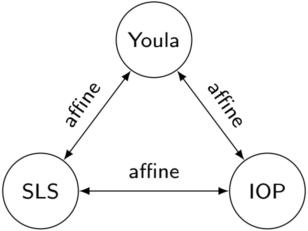
- **Convex reformulation in Youla**

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \left\|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\right\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty, \end{aligned}$$

  where $\mathbf{T}_{11} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{V}_r\mathbf{M}_l\mathbf{P}_{21}, \mathbf{T}_{12} = -\mathbf{P}_{12}\mathbf{M}_r$, and $\mathbf{T}_{21} = \mathbf{M}_l\mathbf{P}_{21}$.

Explicit equivalence among Youla, SLS, and IOP

— any convex SLS can be equivalently reformulated into a convex problem in Youla or IOP; vice versa



- Y. Zheng, L. Furieri, A. Papachristodoulou, N. Li, and M. Kamgarpour. Onthe equivalence of youla, system-level and input-output parameterizations. IEEE Transactions on Automatic Control, 2020.

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Hardy spaces: $\mathcal{H}_2$ and $\mathcal{H}_\infty$

- $\mathcal{L}_2(j\mathbb{R})$ **Space:** this space consists of all complex matrix functions F

  $$\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega < \infty. \qquad F_1(s) = \frac{1}{s-1}, F_2(s) = \frac{1}{s+1}$$

  The inner product is defined as

  $$\langle F, G \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)G(j\omega)\right] d\omega,$$

  for $F, G \in \mathcal{L}_2$, and the induced norm is given by $\|F\|_2 := \sqrt{\langle F, F \rangle}$.
- $\mathcal{H}_2$ **Space:** a subspace of $\mathcal{L}_2$ with matrix functions $F(s)$ analytic in $\text{Re}(s) > 0$. The corresponding norm is defined as

  $$\begin{aligned}\|F\|_2^2 :=& \sup_{\sigma>0}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(\sigma+j\omega)F(\sigma+j\omega)\right] d\omega\right\}.\\ =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\left[F^*(j\omega)F(j\omega)\right] d\omega.\end{aligned}$$

- $\mathcal{RH}_2$ **Space:** The real rational subspace of $\mathcal{H}_2$, consisting of all strictly proper and real rational stable transfer matrices.

Hardy spaces: $\mathcal{H}_2$ and $\mathcal{H}_\infty$

- $\mathcal{L}_\infty(j\mathbb{R})$ **Space**: consisting of matrix-valued complex functions that are bounded on $j\mathbb{R}$, with norm defined as

$$\|F\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}[F(j\omega)]. \qquad F_1(s) = \frac{1}{s-1}, F_2(s) = \frac{1}{s+1}$$

- $\mathcal{H}_\infty$ **Space**: $\mathcal{H}_\infty$ is a subspace of $\mathcal{L}_\infty$ with functions that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty := \sup_{\mathrm{Re}(s)>0} \sigma_{\max}(F(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(F(j\omega)).$$

  The second equality can be regarded as a generalization of the maximum modulus theorem for matrix functions.

- $\mathcal{RH}_\infty$ **Space:** The real rational subspace of $\mathcal{H}_\infty$, consisting of all proper and real rational stable transfer matrices.

$$\mathbf{T}(s) = C(sI - A)^{-1}B + D$$

  with A stable.

Computations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Lemma

Consider a transfer matrix $\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ *with* A *stable. Then, we have*

$$\|\mathbf{T}\|_{\mathcal{H}_2}^2 = \mathit{Trace}(B^\mathsf{T} Q B), \qquad \text{where } A^\mathsf{T} Q + QA + C^\mathsf{T} C = 0,$$
$$\|\mathbf{T}\|_{\mathcal{H}_2}^2 = \mathit{Trace}(C P C^\mathsf{T}), \qquad \text{where } AP + PA^\mathsf{T} + BB^\mathsf{T} = 0.$$

where Q *and* P *are observability and controllability Gramians.*

- **Deterministic interpretation:** Squared $\mathcal{H}_2$ norm is energy sum of transients of output responses:

$$\sum_{k=1}^{m} \int_0^\infty z_k(t)^\mathsf{T} z_k(t) dt = \int_0^\infty \mathrm{Trace}\left((Ce^{At}B)^\mathsf{T} (Ce^{At}B) \right) dt = \|\mathbf{T}\|_{\mathcal{H}_2}^2.$$

- **Stochastic interpretation:** If w is white noise and $\dot{x} = Ax + Bw, z = Cx$

$$\lim_{t\to\infty} \mathbb{E}\left(z(t)^\mathsf{T} z(t) \right) = \|\mathbf{T}\|_{\mathcal{H}_2}^2$$

The squared $\mathcal{H}_2$-norm equals the asymptotic variance of output.

Computations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Lemma

Consider a transfer matrix $\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ *with* A *stable. Then, we have* $\|\mathbf{T}(s)\|_2 < \gamma$ *if and only if there exists* $P \succ 0$ *such that*

$$\text{trace}(CPC^\mathsf{T}) < \gamma^2, \quad \text{and} \quad AP + PA^\mathsf{T} + BB^\mathsf{T} \prec 0,$$

and there exists $Q \succ 0$ *such that*

$$\text{trace}(B^\mathsf{T}QB) < \gamma^2, \quad \text{and} \quad A^\mathsf{T}Q + QA + C^\mathsf{T}C \prec 0.$$

$\Rightarrow$: if $\|\mathbf{T}(s)\|_2 < \gamma$, then we have

$$\text{Trace}(CP_0C^\mathsf{T}) < \gamma^2, \qquad \text{where} \qquad AP_0 + P_0A^\mathsf{T} + BB^\mathsf{T} = 0.$$

Now we consider $AP_\epsilon + P_\epsilon A^\mathsf{T} + BB^\mathsf{T} + \epsilon I = 0$. Note that $\lim_{\epsilon\to 0} P_\epsilon = P_0$. Since $\text{Trace}(CP_0C^\mathsf{T}) < \gamma^2$, there exists a $\epsilon > 0$ such that $\text{Trace}(CP_\epsilon C^\mathsf{T}) < \gamma^2$ and

$$AP_\epsilon + P_\epsilon A^\mathsf{T} + BB^\mathsf{T} = -\epsilon I \prec 0.$$

Computations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

Lemma

Consider a transfer matrix $\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ *with* A *stable. Then, we have* $\|\mathbf{T}(s)\|_2 < \gamma$ *if and only if there exists* $P \succ 0$ *such that*

$$\textit{trace}(CPC^\mathsf{T}) < \gamma^2, \quad \textit{and} \quad AP + PA^\mathsf{T} + BB^\mathsf{T} \prec 0,$$

and there exists $Q \succ 0$ *such that*

$$\textit{trace}(B^\mathsf{T}QB) < \gamma^2, \quad \textit{and} \quad A^\mathsf{T}Q + QA + C^\mathsf{T}C \prec 0.$$

$\Leftarrow$ We first have

$$AP + PA^\mathsf{T} + BB^\mathsf{T} - (AP_0 + P_0A^\mathsf{T} + BB^\mathsf{T}) = A(P - P_0) + (P - P_0)A^\mathsf{T} \prec 0.$$

This indicates that $P - P_0 \succ 0$ (since A is stable). Then

$$\text{trace}(CP_0C^\mathsf{T}) < \text{trace}(CPC^\mathsf{T}) < \gamma^2.$$

We have proved that $\|G(s)\|_2 < \gamma$.

Computations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

We have a special version of *Kalman-Yakubovich-Popov* (KYP) lemma:

Lemma

Consider a transfer matrix $\mathbf{T}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *with* A *stable. Then, the following statements are equivalent:*

- $\|\mathbf{T}(s)\|_\infty < \gamma$;
- $\mathbf{T}^*(j\omega)\mathbf{T}(j\omega) \prec \gamma^2 I, \forall \omega \in \mathbb{R}$.
- *The following LMI is feasible.*

$$\begin{bmatrix} A^\mathsf{T} X + XA & XB & C^\mathsf{T} \\ B^\mathsf{T} X & -\gamma I & D^\mathsf{T} \\ C & D & -\gamma I \end{bmatrix} \prec 0, X \succ 0.$$

- *We have*

$$\sup_{0<\|d\|_2<1} \frac{\|\mathbf{T}(s)d\|_2}{\|d\|_2} < \gamma$$

Other equivalent formulations

The $\mathcal{H}_\infty$ LMI has multiple equivalent forms:

- (obtained by left- and right- multiplied by $\text{diag}(\gamma^{\frac{1}{2}}I, \gamma^{\frac{1}{2}}I, \gamma^{-\frac{1}{2}}I)$)

$$\begin{bmatrix} A^\mathsf{T}X + XA & XB & C^\mathsf{T} \\ B^\mathsf{T}X & -\gamma^2 I & D^\mathsf{T} \\ C & D & -I \end{bmatrix} \prec 0, X \succ 0.$$

- and (by applying the Schur complement)

$$\begin{bmatrix} A^\mathsf{T}X + XA + C^\mathsf{T}C & XB + C^\mathsf{T}D \\ B^\mathsf{T}X + D^\mathsf{T}C & D^\mathsf{T}D - \gamma^2 I \end{bmatrix} \prec 0, \qquad X \succ 0,$$

- and

$$A^\mathsf{T}X + XA + C^\mathsf{T}C - (XB + C^\mathsf{T}D)(D^\mathsf{T}D - \gamma^2 I)^{-1}(B^\mathsf{T}X + D^\mathsf{T}C) \prec 0, \qquad X \succ 0,$$

which is a Riccati inequality.

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State Feedback

- Optimal control

$$\min \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array} \right] \right\|$$

$$\text{s.t.} \quad \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.}$$

- We consider static state feedback $u = D_k x$, and the controller synthesis problem becomes

$$\begin{aligned} \min_{D_k} \quad & \left\| \left[\begin{array}{c|c} A + B_2 D_k & B_1 \\ \hline C_1 + D_{12} D_k & D_{11} \end{array} \right] \right\| \\ \text{subject to} \quad & A + B_2 D_k \text{ is stable.} \end{aligned} \tag{3}$$

LMI for $\mathcal{H}_2$ optimal control

Minimize the $\mathcal{H}_2$ norm of the closed-loop system $\mathbf{T}_{zw}$ and assume $D_{11} = 0$ (otherwise $\mathbf{T}_{zw}$ is not strictly proper and $\|\mathbf{T}_{zw}\|_2$ is not finite).

- Step 1: applying the LMI condition for $\mathcal{H}_2$ norm.

$$\begin{aligned} \min_{P, D_k, \gamma} \quad & \gamma \\ \text{subject to} \quad & (A + B_2 D_k)P + P(A + B_2 D_k)^{\mathsf{T}} + B_1 B_1^{\mathsf{T}} \prec 0, \\ & \text{trace}((C_1 + D_{12} D_k)P(C_1 + D_{12} D_k)^{\mathsf{T}}) < \gamma, \\ & P \succ 0. \end{aligned}$$

- Step 2: Change of variable and introduce $X = D_k P$

$$\begin{aligned} \min_{P, X, \gamma} \quad & \gamma \\ \text{subject to} \quad & (AP + B_2 X) + (AP + B_2 X)^{\mathsf{T}} + B_1 B_1^{\mathsf{T}} \prec 0, \\ & \text{trace}((C_1 P + D_{12} X)P^{-1}(C_1 P + D_{12} X)^{\mathsf{T}}) < \gamma, \\ & P \succ 0. \end{aligned}$$

LMI for $\mathcal{H}_2$ optimal control

- Step 3: Apply the Schur complement, and note that

$$\text{trace}((C_1 P + D_{12} X) P^{-1} (C_1 P + D_{12} X)^\mathsf{T}) < \gamma, \quad P \succ 0$$

is equivalent to

$$\begin{bmatrix} Z & C_1 P + D_{12} X \\ (C_1 P + D_{12} X)^\mathsf{T} & P \end{bmatrix} \succ 0, \quad \text{trace}(Z) < \gamma.$$

- Step 4: get an LMI formulation

$$\begin{aligned} \min_{P,X,Z} \quad & \text{trace}(Z) \\ \text{subject to} \quad & (AP + B_2 X) + (AP + B_2 X)^\mathsf{T} + B_1 B_1^\mathsf{T} \prec 0, \\ & \begin{bmatrix} Z & C_1 P + D_{12} X \\ (C_1 P + D_{12} X)^\mathsf{T} & P \end{bmatrix} \succ 0, \end{aligned}$$

and the optimal $\mathcal{H}_2$ optimal state feedback gain is recovered by $D_k = XP^{-1}$.

LMI formulation for $\mathcal{H}_\infty$ optimal control

Minimize $\|\mathbf{T}_{zw}\|_\infty$ in the optimal control formulation.

- Step 1: apply the LMI for $\mathcal{H}_\infty$ norm

$$
\begin{aligned}
\min_{X, D_k, \gamma} \quad & \gamma \\
\text{subject to} \quad & \begin{bmatrix} (A + B_2 D_k)^\top X + X(A + B_2 D_k) & X B_1 & (C_1 + D_{12} D_k)^\top \\ B_1^\top X & -\gamma I & D_{11}^\top \\ C_1 + D_{12} D_k & D_{11} & -\gamma I \end{bmatrix} \prec 0, \\
& X \succ 0.
\end{aligned}
$$

- Step 2: Left- and right-multiplied by diag(X^{-1}, I, I).

$$
\begin{aligned}
\min_{P, D_k, \gamma} \quad & \gamma \\
\text{subject to} \quad & \begin{bmatrix} P(A + B_2 D_k)^\top + (A + B_2 D_k) P & B_1 & P(C_1 + D_{12} D_k)^\top \\ B_1^\top & -\gamma I & D_{11}^\top \\ (C_1 + D_{12} D_k) P & D_{11} & -\gamma I \end{bmatrix} \prec 0, \\
& P \succ 0,
\end{aligned}
$$

LMI formulation for $\mathcal{H}_\infty$ optimal control

Minimize $\|\mathbf{T}_{zw}\|_\infty$ in the optimal control formulation.

- Step 3: Change of variables $Y = D_k P$.

$$
\begin{aligned}
\min_{P,Y,\gamma} \quad & \gamma \\
\text{subject to} \quad & \begin{bmatrix} (AP + B_2Y)^\mathsf{T} + (AP + B_2Y) & B_1 & (C_1 + D_{12}Y)^\mathsf{T} \\ B_1^\mathsf{T} & -\gamma I & D_{11}^\mathsf{T} \\ (C_1 + D_{12}Y) & D_{11} & -\gamma I \end{bmatrix} \prec 0, \\
& P \succ 0.
\end{aligned}
$$

The optimal $\mathcal{H}_\infty$ state feedback gain can be recovered by $D_k = YP^{-1}$.

General $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Output feedback

- Optimal control

$$\min \left\| \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array}\right] \right\|$$

$$\text{s.t.} \begin{bmatrix} A + B_2 D_k C_2 & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \text{ is stable.}$$

- You can apply the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ LMI and use a series of changes of variables to derive LMI formulations.

1. Pascal Gahinet and Pierre Apkarian. A linear matrix inequality approach to $\mathcal{H}_\infty$ control. Inter-national journal of robust and nonlinear control, 4(4):421–448, 1994.
2. Carsten Scherer, Pascal Gahinet, and Mahmoud Chilali. Multiobjective output-feedback controlvia lmi optimization.IEEE Transactions on automatic control, 42(7):896–911, 1997

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Classical distributed control formulation

- A canonical problem is to minimize a norm of the closed-loop map subject to a subspace constraint

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \in \mathcal{C}_{\text{stab}}, \\ & \mathbf{K} \in \mathcal{S}, \end{aligned} \tag{4}$$

  where $\mathcal{S}$ is a subspace demoting sparsity or delay constraints on the controller.

- After applying the change of variables in Youla, input-output, or system-level parameterization, we need to introduce the following non-convex constraint on the decision variables

$$\begin{aligned} (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} &\in \mathcal{S}, \\ \mathbf{U}\mathbf{Y}^{-1} &\in \mathcal{S}, \\ \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} &\in \mathcal{S}. \end{aligned}$$

Quadratic Invariance

Definition (Quadratic Invariance (QI))

Given a plant $\mathbf{P}_{22}$ and a subspace $\mathcal{S}$. The subspace $\mathcal{S}$ is called **quadratically invariant** under $\mathbf{P}_{22}$ if

$$\mathbf{K}\mathbf{P}_{22}\mathbf{K} \in \mathcal{S}, \qquad \forall \mathbf{K} \in \mathcal{S}.$$

Cayley–Hamilton theorem

$$p(A) = c_0 + c_1 A + \ldots + c_n A^n = 0.$$

Theorem

Define $\mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}$*. If* $\mathcal{S}$ *is quadratically invariant under* $\mathbf{P}_{22}$*, then*

$$\mathbf{K} \in \mathcal{S} \Longleftrightarrow \mathbf{U} \in \mathcal{S}.$$

$\Rightarrow$ Observe that

$$(I - \mathbf{P}_{22}\mathbf{K})^{-1} = \alpha_0 + \alpha_1(I - \mathbf{P}_{22}\mathbf{K}) + \ldots + \alpha_{m-1}(I - \mathbf{P}_{22}\mathbf{K})^{m-1}$$

$\Leftarrow$ Observe that $\mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}$ leads to $\mathbf{K} = \mathbf{U}(I + \mathbf{P}_{22}\mathbf{U})^{-1}$.

QI with the IOP

Theorem (QI with the IOP)

If $\mathcal{S}$ is QI under $\mathbf{P}_{22}$, then

1. *We have*

$$\mathcal{C}_{stab} \cap \mathcal{S} = \{\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \mid \mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z} \text{ are in } (1a)\text{-}(1c), \mathbf{U} \in \mathcal{S}\}.$$

2. *Problem (4) can be equivalently formulated as a convex problem*

$$\begin{aligned} \min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (1a) - (1c), \\ & \mathbf{U} \in \mathcal{S}. \end{aligned}$$

Proof: From the affine constraint, we have

$$\mathbf{Y} - \mathbf{P}_{22}\mathbf{U} = I \quad \Rightarrow \quad \mathbf{Y} = I + \mathbf{P}_{22}\mathbf{U}.$$

Then, we have

$$\mathbf{K} = \mathbf{U}(I + \mathbf{P}_{22}\mathbf{U})^{-1} \quad \Leftrightarrow \quad \mathbf{U} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}.$$

QI with the SLP

Corollary (QI with the SLP)

If $\mathcal{S}$ is QI under $\mathbf{P}_{22}$, then

1. *We have*

$$\mathcal{C}_{stab} \cap \mathcal{S} = \{\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R},\, \mathbf{M},\, \mathbf{N},\, \mathbf{L} \text{ are in } (2a)\text{-}(2c), \mathbf{L} \in \mathcal{S}\}.$$

2. *Problem (4) can be equivalently formulated as a convex problem*

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & (2a) - (2c), \\ & \mathbf{L} \in \mathcal{S}. \end{aligned}$$

QI with the Youla

Corollary (QI with Youla)

If $\mathcal{S}$ is QI under $\mathbf{P}_{22}$, then

1. *We have*

$$\mathcal{C}_{stab} \cap \mathcal{S} = \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty \\ (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l \in \mathcal{S}, \}.$$

2. *Problem (4) can be equivalently formulated as a convex problem*

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \textit{subject to} \quad & (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l \in \mathcal{S} \\ & \mathbf{Q} \in \mathcal{RH}_\infty. \end{aligned}$$

Summary: Quadratic invariance (QI)

Youla

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty, \\ & (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \in \mathcal{S} \end{aligned}$$

$$(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l \in \mathcal{S}$$

IOP

$$\begin{aligned} \min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{U}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (1a) - (1c). \\ & \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{S} \end{aligned}$$

$$\mathbf{U} \in \mathcal{S}$$

SLS

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{22} \right\| \\ \text{subject to} \quad & (2a) - (2c) \\ & \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \in \mathcal{S}. \end{aligned}$$

$$\mathbf{L} \in \mathcal{S}$$

Other topics

- QI does not necessarily promise efficient numerical computation; Many later works aim to provide state-space solutions.
- Sparsity Invariance: Beyond QI for sparsity constraints;

$$\mathbf{U} \in \mathcal{T}, \mathbf{Y} \in \mathcal{R} \qquad \Rightarrow \qquad \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{S}$$

- Youla for distributed control: Gradient dominance and its connections with learning applications.

Hardy spaces: $\mathcal{H}_2$ and $\mathcal{H}_\infty$

- **Complex function:** Given $S \subset \mathbb{C}$, define $f(s)$ as a complex valued function on S:

$$f(s) : S \to \mathbb{C}.$$

- **Analytical complex function:** $f(s)$ is said to be analytic at a point z_0 in S if it is differentiable at z_0 and also at each point in some neighborhood of z_0.

$$\lim_{s \to z_0} \frac{f(s) - f(z_0)}{s - z_0}.$$

  A function $f(s)$ analytic at z_0 has a power series representation at z_0, *i.e.*,

$$f(s) = c_0 + \sum_{n=1}^{\infty} c_n (s - z_0)^n,$$

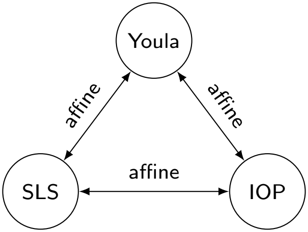
  converges for some neighborhood of z_0.

- **Analytic complex function matrix:** A matrix valued function is analytic in S if every element of the matrix is analytic in S.

Appendix

Explicit equivalence among Youla, SLS, and IOP

— any convex SLS can be equivalently reformulated into a convex problem in Youla or IOP; vice versa



Youla $\Leftrightarrow$ IOP

Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{G}$. We have

1. For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\begin{aligned}
\mathbf{Y} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l\,,\\
\mathbf{U} &= (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l\,,\\
\mathbf{W} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{N}_l\,,\\
\mathbf{Z} &= I + (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{N}_l\,,
\end{aligned}$$

   belong to the IOP constraint and are such that

$$\mathbf{U}\mathbf{Y}^{-1} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}.$$

2. For any $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$ in the IOP constraint, the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l\mathbf{Y}\mathbf{U}_r - \mathbf{U}_l\mathbf{U}\mathbf{U}_r - \mathbf{V}_l\mathbf{W}\mathbf{V}_r + \mathbf{U}_l\mathbf{Z}\mathbf{V}_r - \mathbf{V}_l\mathbf{U}_r\,,$$

   is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} = \mathbf{U}\mathbf{Y}^{-1}$.

IOP $\Leftrightarrow$ SLS

For any $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ satisfying the SLP constraint, the transfer matrices

$$\begin{aligned} \mathbf{Y} &= C_2\mathbf{N} + I, \\ \mathbf{U} &= \mathbf{L}, \\ \mathbf{W} &= C_2\mathbf{R}B_2, \\ \mathbf{Z} &= \mathbf{M}B_2 + I, \end{aligned}$$

belong to the IOP constraint and are such that

$$\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} = \mathbf{U}\mathbf{Y}^{-1}.$$

- The affine relationship can written into

$$\begin{bmatrix} \mathbf{Y} & \mathbf{W} \\ \mathbf{U} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} C_2 & \\ & I \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} & B_2 \\ I & \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

- *This affine transformation is in general not invertible, but considering the achievability conditions, an explicit converse transformation can be found as well.*

IOP $\Leftrightarrow$ SLS

For any $\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}$ satisfying the IOP constraint, the transfer matrices

$$\begin{aligned} \mathbf{R} &= (sI - A)^{-1} + (sI - A)^{-1} B_2 \mathbf{U} C_2 (sI - A)^{-1} \\ \mathbf{M} &= \mathbf{U} C_2 (sI - A)^{-1}, \\ \mathbf{N} &= (sI - A)^{-1} B_2 \mathbf{U}, \\ \mathbf{L} &= \mathbf{U}, \end{aligned}$$

belong to the SLP constraint and are such that

$$\mathbf{U}\mathbf{Y}^{-1} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}.$$

Youla $\Leftrightarrow$ SLS

Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{G}$. We have

1. For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\begin{aligned}
\mathbf{R} &= (sI - A)^{-1} + (sI - A)^{-1} B_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l C_2 (sI - A)^{-1} \\
\mathbf{M} &= (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l C_2 (sI - A)^{-1}, \\
\mathbf{N} &= (sI - A)^{-1} B_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \\
\mathbf{L} &= (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l,
\end{aligned}$$

   belong to the SLP constraint and are such that

$$\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1}.$$

2. For any $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ in the SLP constraint, the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l C_2 \mathbf{N} \mathbf{U}_r - \mathbf{U}_l \mathbf{L} \mathbf{U}_r - \mathbf{V}_l C_2 \mathbf{R} B_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{M} B_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{V}_r$$

   is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$.

Youla $\Leftrightarrow$ SLS $\Leftrightarrow$ IOP

Convex system-level synthesis: (Wang et al., 2019)

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & g(\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}) \\ \text{subject to} \quad & \text{SLP constraint}, \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

- This is clearly equivalent to a convex problem in Youla,

$$\begin{aligned} \min_{\mathbf{Q}} \quad & g_1(\mathbf{Q}) \\ \text{subject to} \quad & \begin{bmatrix} f_1(\mathbf{Q}) & f_3(\mathbf{Q}) \\ f_2(\mathbf{Q}) & f_4(\mathbf{Q}) \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

- which is also equivalent to a convex problem in input-output parameterization

$$\begin{aligned} \min_{\mathbf{Y},\mathbf{U},\mathbf{W},\mathbf{Z}} \quad & \hat{g}_1(\mathbf{U}) \\ \text{subject to} \quad & \text{IOP constraint} \\ & \begin{bmatrix} \hat{f}_1(\mathbf{U}) & \hat{f}_3(\mathbf{U}) \\ \hat{f}_2(\mathbf{U}) & \hat{f}_4(\mathbf{U}) \end{bmatrix} \in \mathcal{S}. \end{aligned}$$