Sparsity Invariance for Convex Design of Distributed Controllers

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Abstract

We address the problem of designing optimal distributed controllers for linear time invariant (LTI) systems, which corresponds to minimizing a norm of the closed-loop system subject to sparsity constraints on the controller structure. This problem is NP-hard in general and motivates the development of tractable approximations. We characterize a class of convex restrictions based on a new notion of Sparsity Invariance (SI). The underlying idea of SI is to design sparsity patterns for transfer matrices $\mathbf{Y}(s)$ and $\mathbf{X}(s)$ such that any corresponding controller $\mathbf{K}(s) = \mathbf{Y}(s)\mathbf{X}(s)^{-1}$ exhibits the desired sparsity pattern. For sparsity constraints, the approach of SI goes beyond the well-known notion of Quadratic Invariance (QI) in the sense that 1) the SI framework returns a convex restriction for any distributed control problem independently of whether QI holds; 2) the solution via the SI approach is guaranteed to be globally optimal when QI holds and performs at least as well as that obtained by considering a nearest QI subset. Moreover, the notion of SI can be naturally applied to the problem of designing structured static state-feedback controllers, while QI is not utilizable. Numerical examples show that even for non-QI cases, SI can recover solutions that are 1) globally optimal and 2) strictly more performing than previous methods.

1 Introduction

The safe and efficient operation of several large-scale systems, such as the smart grid [1], biological networks [2], and automated highways [3], relies on the decision making of multiple interacting agents. Coordinating the decisions of these agents is challenged by a lack of complete information of the systems' internal variables. Such limited information arises due to privacy concerns, geographic distance or the challenges of implementing a reliable communication network. The celebrated work of [4] highlighted that lacking full information can enormously complicate the design of optimal control inputs. Indeed, the optimal feedback control policies may not even be linear for the Linear Quadratic Gaussian (LQG) control problem. The intractability inherent to lack of full information was investigated in the works [5, 6]. The core challenges discussed therein motivated identifying special cases of optimal control problems with partial information for which efficient algorithms can be used.

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Optimally controlling a linear time-invariant system (LTI) with distributed sensor measurements amounts to computing a linear controller that complies with a desired sparsity pattern and minimizes a norm of the closed-loop system. For this generally intractable problem, the notion of Quadratic Invariance (QI) was shown to be necessary and sufficient for an exact convex reformulation [7]. The interested reader is referred to [8] for an overview on QI and related information structures.

1.1 Previous work on non-QI cases

Given the importance and intricacy of computing optimal distributed controllers, a variety of approximation methods have been proposed for general systems and information structures that are not QI. For example, the authors in [9] developed semidefinite programs (SDP) that are relaxations of this generally NP-hard problem. However, these relaxations might fail to recover a sparse controller that is stabilizing, as confirmed experimentally in [10]. To address this issue, polynomial optimization has been used in [10] to obtain a sequence of convex relaxations which converges to a stabilizing distributed controller. Nevertheless, performance of the recovered solution is not directly addressed in [10]. For the finite-horizon control problem, [11] derived convex upper bounds to the non-convex cost function to obtain conservative feasible solutions. However, the theoretical suboptimality bounds were shown to be loose. Alternatively, the system level approach [12] proposed an implementation where controllers are required to share their past control signals. We note that the classical distributed control only requires to share output measurements, but no control signals, among subsystems. The need to share past control inputs in [12] might raise concerns of system security and vulnerability in safety critical applications [13], where each subsystem can only rely on its own sensor measurements.

A different approach to sparse output-feedback controller synthesis is to develop a *convex restriction*: the unstructured problem is reformulated as an equivalent convex program and convex constraints are added to guarantee the desired sparsity pattern of the recovered controllers. Convex restrictions exhibit specific advantages: 1) their optimal solutions can be readily computed with standard convex optimization techniques, and 2) all their feasible solutions are structured and stabilizing by design. A disadvantage is that a restriction may be infeasible even when the original problem is feasible. This motivates developing convex restrictions that are as tight as possible for improved feasibility and performance. In the literature, convex restrictions have mostly been developed for the special case of computing static controllers [14–16]. For the general case of dynamic controllers given non-QI information structures, the work [17] suggested restricting the desired sparsity pattern to a subset that is QI and thus obtain upper bounds on the minimum cost. However, to the best of the authors' knowledge, a method for convex restrictions that can outpeform [17] and goes beyond the notion of QI is not known.

1.2 Contributions

This paper proposes a generalized framework for the convex design of optimal and near-optimal dynamic output-feedback controllers with a pre-determined sparsity pattern. Our underlying idea is to identify appropriate sparsity patterns for two transfer matrices $\mathbf{Y}(s)$ and $\mathbf{X}(s)$ such that any corresponding feedback controller in the form $\mathbf{K}(s) = \mathbf{Y}(s)\mathbf{X}(s)^{-1}$ exhibits the desired structure. This fundamental property is denoted as Sparsity Invariance (SI).

Our first contribution is to develop algebraic conditions on the binary matrices associated with the sparsities of $\mathbf{Y}(s)$ and $\mathbf{X}(s)$ that are necessary and sufficient for SI. Among all such sparsities, we suggest a polynomial-time algorithm to design sparsities that lead to better performance for the distributed control problem at hand. Second, we show that the SI framework steps beyond the QI notion in several ways. Indeed, SI can be applied to systems with any information structure regardless of whether QI holds. Furthermore, SI recovers a controller that is provably globally optimal when QI holds and performs at least as well as that obtained by considering a nearest QI subset [17] when QI does not hold. Third, we provide examples to show that, even if QI does not hold, controllers obtained through the SI approach can be 1) globally optimal and 2) in general strictly more performing than those obtained using the nearest QI subset approach of [17]. Finally, we remark that the SI framework is directly applicable to distributed static controller design, as studied in our preliminary work [16], whereas the Youla parametrization and thus the QI notion is not utilizable. For clarity, we mainly focus on continuous-time systems in the paper, but the results can be naturally extended to discrete-time systems.

The rest of this paper is structured as follows. Section 2 states necessary background and formulates the problem. Section 3 introduces the class of convex restrictions under investigation and fully characterizes our notion of Sparsity Invariance (SI). We describe how SI can be utilized in an optimized way. In Section 4, we show that 1) SI encompasses the previous approaches based on the QI notion, and 2) that strictly better performing distributed controllers can be computed efficiently with the SI framework. We present numerical results in Section 5 and conclude the paper in Section 6.

2 Background and Problem Statement

Here, we first introduce some notation on sparsity structures and transfer functions. Then, we state the problem of distributed optimal control, and introduce the necessary background on the Youla parametrization of internally stabilizing controllers.

2.1 Notation and sparsity structures

We use $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ to denote real numbers, complex numbers and positive integers, respectively. The (i,j)-th element in a matrix $Y \in \mathbb{R}^{m\times n}$ is referred to as Y_{ij}. We use I_n to denote the identity matrix of size $n \times n$, $0_{m\times n}$ to denote the zero matrix of size $m \times n$ and $1_{m\times n}$ to denote the matrix of size $m \times n$ with all entries set to 1.

Transfer functions: We denote the imaginary axis as $j\mathbb{R} := \{z \in \mathbb{C} \mid \Re(z) = 0\}$. We consider continuous-time transfer functions, defined as rational functions $\mathbf{g} : j\mathbb{R} \to \mathbb{C}$. A transfer function is called *proper* (resp. *strictly-proper*) if the degree of the numerator polynomial does not exceed (resp. is strictly lower than) the degree of the denominator polynomial. We define the *poles* of $\mathbf{g}$ as the roots of the denominator polynomials of $\mathbf{g}$. Similar to [7], we denote by $\mathcal{R}_p^{m\times n}$ the set of $m \times n$ proper *transfer matrices*, that is the set of $m \times n$ matrices whose entries are proper transfer functions. Also, we denote by $\mathcal{R}_{sp}^{m\times n}$ the set of $m \times n$ strictly proper transfer matrices. Finally, we let $\mathcal{RH}_\infty^{m\times n}$ be the set of $m \times n$ proper *stable* transfer matrices, i.e.,

$$\mathcal{RH}_\infty^{m\times n} := \{\mathbf{G} \in \mathcal{R}_p^{m\times n} \mid \mathbf{G} \text{ has no poles in } \mathbb{C}_+\}\,.$$

where $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \Re(z) \geq 0\}$. Sparsity structures of transfer matrices can be conveniently represented by binary matrices. A binary matrix is a matrix with entries from the set $\{0,1\}$, and we use $\{0,1\}^{m\times n}$ to denote the set of $m\times n$ binary matrices. Given a binary matrix $X \in \{0,1\}^{m\times n}$, we define the associated *sparsity subspace* of transfer matrices as

$$\begin{aligned}\text{Sparse}(X) := \{\mathbf{Y} \in \mathcal{R}_p^{m\times n} \mid \mathbf{Y}_{ij}(j\omega) = 0 \;\text{ for all } i,j \\ \text{such that } X_{ij} = 0 \text{ for almost all } \omega \in \mathbb{R}\}\,.\end{aligned}$$

Similarly, given a transfer function $\mathbf{Y} \in \mathcal{R}_p^{m\times n}$, we define $X = \text{Struct}(\mathbf{Y})$ as the binary matrix given by

$$X_{ij} := \begin{cases} 0 & \text{if } \mathbf{Y}_{ij}(j\omega) = 0 \text{ for almost all } \omega \in \mathbb{R}, \\ 1 & \text{otherwise}. \end{cases}$$

We say that the transfer matrix $\mathbf{G} \in \mathcal{R}_p^{n\times n}$ is invertible if $\mathbf{G}(j\omega) \in \mathbb{C}^{n\times n}$ is invertible for almost all $\omega \in \mathbb{R}$.

Let $X, \hat{X} \in \{0,1\}^{m\times n}$ and $Z \in \{0,1\}^{n\times p}$ be binary matrices. Throughout the paper, we adopt the following conventions: $X + \hat{X} := \text{Struct}(X + \hat{X})$, and $XZ := \text{Struct}(XZ)$. We say $X \leq \hat{X}$ if and only if $X_{ij} \leq \hat{X}_{ij}$ $\forall i, j$, and $X < \hat{X}$ if and only if $X \leq \hat{X}$ and there exist indices i, j such that $X_{ij} < \hat{X}_{ij}$. Also, we denote $X \nleq \hat{X}$ if and only if there exist indices i, j such that $X_{ij} > \hat{X}_{ij}$. Given a binary matrix $X \in \{0,1\}^{m\times n}$ we denote its cardinality, *i.e.*, the total number of nonzero entries, as

$$\|X\|_0 := \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}.$$

Considering the following binary matrices

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

we have $X_2 < X_1, X_3 \nleq X_1$ and $X_2 + X_1 = X_1$. Their cardinalities are $\|X_1\|_0 = 4, \|X_2\|_0 = 3$ and $\|X_3\|_0 = 4$, respectively. For the following transfer matrix,

$$\mathbf{Y} = \begin{bmatrix} 0 & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix} \in \mathcal{RH}_\infty^{2\times 3},$$

if we consider the binary matrix X_1 in the example above, we have $\mathbf{Y} \in \text{Sparse}(X_1)$ and $X_1 = \text{Struct}(\mathbf{Y})$.

2.2 Problem statement

We consider linear systems in continuous-time

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + H_x w(t), \\ y(t) &= C_y x(t) + H_y w(t), \\ z(t) &= C_z x(t) + D_z u(t) + H_z w(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^q$, and $w(t) \in \mathbb{R}^r$ are the state, control input, observed output, a performance signal defined based on our control objectives, and additive disturbance at time $t \in \mathbb{R}$, respectively. The input-output transfer function representation for (1) can be written as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathbf{P} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{G} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix},$$

with

$$\begin{aligned} \mathbf{P}_{11} &:= C_z(sI_n - A)^{-1}H_x + H_z, \\ \mathbf{P}_{12} &:= C_z(sI_n - A)^{-1}B + D_z, \\ \mathbf{P}_{21} &:= C_y(sI_n - A)^{-1}H_x + H_y, \\ \mathbf{G} &:= C_y(sI_n - A)^{-1}B, \end{aligned}$$

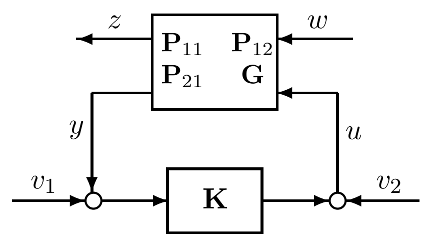

Figure 1: Interconnection of **P** and **K**.

where s belongs to $j\mathbb{R}$. Notice that $\mathbf{P}_{11}, \mathbf{P}_{12}, \mathbf{P}_{21}$ are proper transfer functions and $\mathbf{G}$ is strictly proper.

Consider the interconnection of Figure 1. A dynamic output-feedback controller $u = \mathbf{K}y$ with $\mathbf{K} \in \mathcal{R}_p^{m\times p}$ is said to be *internally stabilizing* if and only if the nine transfer matrices from $w,\ \nu_1,\ \nu_2$ to $z,\ y,\ u$ are stable. We denote the set of all internally stabilizing output-feedback controllers as $\mathcal{C}_{\text{stab}}$. We say that $\mathbf{P}$ is stabilizable if only and if $\mathcal{C}_{\text{stab}} \neq \emptyset$ and any $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ stabilizes $\mathbf{P}$. Furthermore, we say that a controller $\mathbf{K}$ stabilizes $\mathbf{G}$ if and only if the four transfer matrices from $\nu_1,\ \nu_2$ to $y,\ u$ are all stable. For the rest of the paper we make the following assumption.

Assumption 1: The system $\mathbf{P}$ is stabilizable.

A test for stabilizability of $\mathbf{P}$ is offered in [18, Chapter 4]. It is well-known [18, Chapter 4], [7] that under Assumption 1 a controller $\mathbf{K}$ stabilizes $\mathbf{P}$ if and only if it stabilizes $\mathbf{G}$. The control problem is to compute a dynamic output-feedback controller $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ which minimizes a given norm $\|\cdot\|$ of

$$f(\mathbf{K}) = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I_p - \mathbf{G}\mathbf{K})^{-1}\mathbf{P}_{21}\,, \tag{2}$$

which is the closed-loop transfer function from w to z.

In distributed control, we add the requirement that $\mathbf{K}$ only uses partial output measurements. This requirement can be captured by adding the constraint $\mathbf{K} \in \text{Sparse}(S)$ for a given binary matrix $S \in \{0,1\}^{m\times p}$, where $S_{ij} = 0$ encodes the fact that the i-th scalar control input cannot measure the j-th measurement output. We formulate this distributed control problem as follows [7].

Problem $\mathcal{P}_K$

$$\begin{aligned} \underset{\mathbf{K}\in\mathcal{C}_{\text{stab}}}{\text{minimize}} \quad & \|f(\mathbf{K})\| \\ \text{subject to} \quad & \mathbf{K} \in \text{Sparse}(S)\,, \end{aligned}$$

where $\|\cdot\|$ is any norm of interest. It was shown that a necessary and sufficient condition for a feasible solution to $\mathcal{P}_K$ to exist is that all the distributed fixed modes associated with S lie in the left half of the complex plane [19]. Even if $\mathcal{P}_K$ is feasible, directly computing its optimal solution is intractable because the set $\mathcal{C}_{\text{stab}}$ is non-convex in general. This can be easily verified by checking that when $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{C}_{\text{stab}}$, the controller $\mathbf{K} = \frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2)$ does not lie in $\mathcal{C}_{\text{stab}}$ in general. Furthermore, the cost function $\|f(\mathbf{K})\|$ is non-convex in $\mathbf{K}$.

2.3 The Youla parametrization of stabilizing controller

The first step to convexify problem $\mathcal{P}_K$ is to derive a convex formulation of the set $\mathcal{C}_{\text{stab}}$ and the function $f(\mathbf{K})$. This is achieved by using a *doubly coprime factorization* of $\mathbf{G}$.

Lemma 1 (Chapter 4 of [18]) *For any* $\mathbf{G} \in \mathcal{R}_{sp}^{p\times m}$*, there exist eight proper and stable transfer matrices defining a doubly coprime factorization of* $\mathbf{G}$*, that is, they satisfy*

$$
\begin{aligned}
&\mathbf{G} = \mathbf{N}_r\mathbf{M}_r^{-1} = \mathbf{M}_l^{-1}\mathbf{N}_l\,,\\
&\begin{bmatrix}\mathbf{U}_l & -\mathbf{U}_l\\ -\mathbf{N}_l & \mathbf{M}_l\end{bmatrix}\begin{bmatrix}\mathbf{M}_r & \mathbf{V}_r\\ \mathbf{N}_r & \mathbf{U}_r\end{bmatrix} = I_{m+p}\,.
\end{aligned}
\tag{3}
$$

Then, the Youla parametrization of all internally stabilizing controllers [20] establishes the following equivalence [18, Chapter 4]:

$$\mathcal{C}_{\text{stab}} = \{(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}|\ \mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}\}^1\,. \tag{4}$$

In other words, given a doubly-coprime factorization of $\mathbf{G}$, $\mathcal{C}_{\text{stab}}$ can be expressed as an affine map over the linear subspace of stable Youla parameters $\mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}$. Furthermore, it was proved in [18, Chapter 4] that the set of all closed-loop transfer functions from w to z achievable by $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ is

$$f(\mathcal{C}_{\text{stab}}) = \{\mathbf{T}_1 - \mathbf{T}_2\mathbf{Q}\mathbf{T}_3|\ \mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}\}\,,$$

where $f(\cdot)$ is defined in (2) and $\mathbf{T}_1 = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{V}_r\mathbf{M}_l\mathbf{P}_{21}$, $\mathbf{T}_2 = \mathbf{P}_{12}\mathbf{M}_r$ and $\mathbf{T}_3 = \mathbf{M}_l\mathbf{P}_{21}$. To facilitate our problem formulation, we define

$$\mathbf{Y}_Q = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l\,, \tag{5}$$

$$\mathbf{X}_Q = (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l\,. \tag{6}$$

It directly follows from from (4) that

$$\mathcal{C}_{\text{stab}} = \{\mathbf{Y}_Q\mathbf{X}_Q^{-1}|\ (5), (6),\ \mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}\}\,. \tag{7}$$

We notice that (3) implies $\mathbf{U}_r = \mathbf{M}_l^{-1} + \mathbf{G}\mathbf{V}_r$ and (5) implies $\mathbf{V}_r\mathbf{M}_l = \mathbf{Y}_Q + \mathbf{M}_r\mathbf{Q}\mathbf{M}_l$. Hence,

$$
\begin{aligned}
\mathbf{X}_Q &= (\mathbf{M}_l^{-1} + \mathbf{G}\mathbf{V}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l\\
&= I_p + \mathbf{G}(\mathbf{Y}_Q + \mathbf{M}_r\mathbf{Q}\mathbf{M}_l) - \mathbf{N}_r\mathbf{Q}\mathbf{M}_l\\
&= I_p + \mathbf{G}\mathbf{Y}_Q\,.
\end{aligned}
\tag{8}
$$

Now we can equivalently reformulate $\mathcal{P}_K$ into the following optimization problem.

Problem $\mathcal{P}_Q$

$$
\begin{aligned}
&\underset{\mathbf{Q}\in\mathcal{RH}_\infty^{m\times p}}{\text{minimize}} && \|\mathbf{T}_1 - \mathbf{T}_2\mathbf{Q}\mathbf{T}_3\|\\
&\text{subject to} && (5),\quad (6),\quad \mathbf{Y}_Q\mathbf{X}_Q^{-1} \in \text{Sparse}(S)\,.
\end{aligned}
$$

Without the sparsity constraint Sparse(S), problem $\mathcal{P}_Q$ would be convex, as (5), (6) and the cost function are affine in $\mathbf{Q}$. The primary source of non-convexity is the requirement that $\mathbf{Y}_Q\mathbf{X}_Q^{-1} \in$ Sparse(S). We conclude that the complexity of distributed control is ultimately linked to non-convex sparsity requirements on the Youla parameter.

[1] Equivalently, $\mathcal{C}_{\text{stab}} = \{(\mathbf{U}_l - \mathbf{Q}\mathbf{N}_l)^{-1}(\mathbf{V}_l - \mathbf{Q}\mathbf{M}_l)|\ \mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}\}$.

3 Sparsity Invariance

One approach to remove non-convex sparsity requirements on the Youla parameter is as follows: replace the non-convex constraint $\mathbf{Y}_Q\mathbf{X}_Q^{-1} \in \text{Sparse}(S)$ with the convex constraint that $\mathbf{Y}_Q$ and $\mathbf{X}_Q$ comply with appropriate sparsity patterns, in a way such that $\mathbf{Y}_Q\mathbf{X}_Q^{-1}$ is guaranteed to lie in $\text{Sparse}(S)$. In other words, we restrict our attention to distributed controllers $\mathbf{K} \in \text{Sparse}(S)$ defined as the product of two structured matrix factors.

Following the general idea above, in this paper we investigate a notion of Sparsity Invariance (SI) for convex design of distributed controllers. As will be thoroughly discussed in Section 4, SI leads to the largest known class of convex restrictions of $\mathcal{P}_K$ for general systems and information structures.

Definition 1 (Sparsity Invariance (SI)) *Given a binary matrix S, the pair of binary matrices T, R satisfies a property of sparsity invariance (SI) with respect to S if*

$$\begin{gathered} \mathbf{Y} \in \text{Sparse}(T) \text{ and } \mathbf{X} \in \text{Sparse}(R) \\ \Downarrow \\ \mathbf{Y}\mathbf{X}^{-1} \in \text{Sparse}(S). \end{gathered} \tag{9}$$

Motivated by the SI property, consider the following convex problem:

Problem $\mathcal{P}_{T,R}$

$$\begin{aligned} \underset{\mathbf{Q}\in\mathcal{RH}_\infty^{m\times p}}{\text{minimize}} \quad & \|\mathbf{T}_1 - \mathbf{T}_2\mathbf{Q}\mathbf{T}_3\| \\ \text{subject to} \quad & (5),\ (6)\,, \\ & \mathbf{Y}_Q\mathbf{\Gamma} \in \text{Sparse}(T),\ \mathbf{X}_Q\mathbf{\Gamma} \in \text{Sparse}(R)\,, \end{aligned}$$

where $T \in \{0,1\}^{m\times p}, R \in \{0,1\}^{p\times p}$ and $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$, with $\mathbf{\Gamma}$ invertible, are parameters to be designed before performing the optimization. For simplicity, one could select $\mathbf{\Gamma} = I_p$, but we illustrate in Example 1 of Section 4 that there are cases where a different choice of $\mathbf{\Gamma}$ might lead to improved and even globally-optimal performance for non-QI problems. For any choice of T, R and $\mathbf{\Gamma}$, the above program is convex. One fundamental question is when its feasible solutions lead to stabilizing controllers $\mathbf{K} = (\mathbf{Y}_Q\mathbf{\Gamma})(\mathbf{X}_Q\mathbf{\Gamma})^{-1} = \mathbf{Y}_Q\mathbf{X}_Q^{-1}$ lying in the desired sparsity subspace $\text{Sparse}(S)$. The notion of SI (9) defined above is a mathematical expression of this requirement. In the next subsection we establish necessary and sufficient conditions on the binary matrices T and R to satisfy the SI property (9).

Remark 1 We assume that $R \geq I_p$. Since $\mathbf{X}_Q = I_p + \mathbf{G}\mathbf{Y}_Q \in \text{Sparse}(R)$ and $\mathbf{G}$ is strictly proper, the assumption is without loss of generality for $\mathbf{\Gamma} = I_p$. For convenience, in the definition of problem $\mathcal{P}_{T,R}$ we do not indicate $\mathbf{\Gamma}$ explicitly as a parameter. This is because the SI property (9) only depends on the binary matrices T and R.

3.1 Characterization of SI

One immediate idea in designing the binary matrices T and R to guarantee $\mathbf{K} = (\mathbf{Y_Q}\mathbf{\Gamma})(\mathbf{X_Q}\mathbf{\Gamma})^{-1} = \mathbf{Y}_Q\mathbf{X}_Q^{-1} \in \text{Sparse}(S)$ is to simply select $T = S$ and $R = I_p$ similar to [14, 15, 21]. However, many other choices are available that lead to improved convex restrictions.

The next Theorem provides a full characterization of the SI property (9) in terms of the binary matrices T and R.

Theorem 1 *Let $T \in \{0,1\}^{m\times p}$ and $R \in \{0,1\}^{p\times p}$ be such that $R \geq I_p$. The following two statements are equivalent:*

1. *$T \leq S$ and $TR^{p-1} \leq S$.*

2. *SI as per (9) holds.*

The proof of Theorem 1 is reported in the Appendix. The relevance of Theorem 1 to characterizing a class of convex restrictions of $\mathcal{P}_K$ is stated in the following Corollary.

Corollary 1 *Let $T \in \{0,1\}^{m\times p}$ and $R \in \{0,1\}^{p\times p}$ be such that $R \geq I_p$, $T \leq S$ and $TR^{p-1} \leq S$. Then, problem $\mathcal{P}_{T,R^{p-1}}$ is a convex restriction of $\mathcal{P}_K$ for any invertible transfer matrix $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$.*

Proof Problem $\mathcal{P}_{T,R^{p-1}}$ is obviously convex. We only need to show that any solution to $\mathcal{P}_{T,R^{p-1}}$ corresponds to a feasible solution of $\mathcal{P}_Q$. Indeed, for any invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$ we have $(\mathbf{Y}_Q\mathbf{\Gamma})(\mathbf{X}_Q\mathbf{\Gamma})^{-1} = \mathbf{Y}_Q\mathbf{X}_Q^{-1}$. Let $\mathbf{Y} = \mathbf{Y}_Q\mathbf{\Gamma}$ and $\mathbf{X} = \mathbf{X}_Q\mathbf{\Gamma}$ in (9). Since (9) holds by Theorem 1, by definition $\mathbf{Y}\mathbf{X}^{-1} = \mathbf{Y}_Q\mathbf{X}_Q^{-1} \in \text{Sparse}(S)$ and thus every solution of $\mathcal{P}_{T,R}$ is also a solution of $\mathcal{P}_Q$. Since $\mathcal{P}_Q$ is equivalent to $\mathcal{P}_K$, we conclude that $\mathcal{P}_{T,R}$ is a restriction of $\mathcal{P}_K$ for every invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$. Since $TR^{p-1} \leq S$ and $R \geq I_p$ we have that $T(R^{p-1})^{p-1} \leq S$. Hence, $\mathcal{P}_{T,R^{p-1}}$ is a convex restriction of $\mathcal{P}_K$ for every invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$.

We note that Theorem 1 and Corollary 1 directly extend [16, Theorem 1], which was only valid for designing static distributed controllers. In summary, the algebraic conditions

$$T \leq S \text{ and } TR^{p-1} \leq S\,, \tag{10}$$

are equivalent to SI and yield a class of convex restrictions of $\mathcal{P}_K$. Clearly, our condition (10) includes the choice $T = S$ and R is (block)-diagonal as per [14, 15, 21]. We will further show in Section 4 that the convex restrictions developed in [17] are a particular case of (10). Therefore, our notion of SI naturally encompasses and extends previous convex restrictions of $\mathcal{P}_K$.

Remark 2 For each T and R as per (10), it is always preferable to solve the convex restriction $\mathcal{P}_{T,R^{p-1}}$ instead of $\mathcal{P}_{T,R}$. Indeed, notice that since $TR^{p-1} \leq S$ and $R \geq I_p$, then $T(R^{p-1})^{p-1} \leq S$. Equivalently, when T and R satisfy sparsity invariance (10), so do T and R^{p-1}, and both $\mathcal{P}_{T,R}$ and $\mathcal{P}_{T,R^{p-1}}$ are convex restrictions of $\mathcal{P}_K$. Since requiring $\mathbf{X}_Q \in \text{Sparse}(R')$ for some $R' < R^{p-1}$ can be conservative due to $\text{Sparse}(R') \subset \text{Sparse}(R^{p-1})$, we will mainly focus on the convex restriction $\mathcal{P}_{T,R^{p-1}}$ for the rest of the paper.

After determining all the matrices T and R for sparsity invariance, a natural follow-up question arises: how can we choose T and R as per Theorem 1 to obtain a convex restriction of $\mathcal{P}_K$ that is as tight as possible?

3.2 Optimized design of SI

Here, we study how to choose the sparsities T and R optimally for a fixed invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$.

In order to determine the best performing choice for T and R satisfying (10), one would need in general to solve $\mathcal{P}_{T,R^{p-1}}$ with the chosen $\mathbf{\Gamma}$ for each T and R such that (10) holds, and then select the problem minimizing the objective $\|\mathbf{T}_1 - \mathbf{T}_2\mathbf{Q}\mathbf{T}_3\|$. Clearly, this approach is not tractable in general, as one needs to solve a number of convex programs that is exponential in m and p, that is, one convex program for each binary matrices T and R such that $TR^{p-1} \leq S$. Even if we simplify the search above by fixing any $T \leq S$ and looking for the best performing choice of R, we would still need to solve a number of convex programs that is exponential in p, that is, one convex program for each binary matrix R such that $TR^{p-1} \leq S$. To deal with this challenge, we suggest a computationally efficient algorithm that generates an optimized binary matrix R given a fixed $T \leq S$. We build upon our past work [16], where we identified optimized separable Lyapunov functions for designing static state-feedback controllers. A main difference is that here the binary matrix R need not be symmetric and a Lyapunov interpretation is not relevant.

Our suggested approach is to design the binary matrix $R_T^\star$ that yields the tightest convex restriction $\mathcal{P}_{T,R_T^\star}$ of $\mathcal{P}_K$ among all the $\mathcal{P}_{T,R}$'s where $T \leq S$ is fixed and R is any binary matrix satisfying

$$TR^{p-1} \leq T\,. \tag{11}$$

Clearly, $T \leq S$ and (11) together imply (10). Such an $R_T^\star$ can be computed as per Algorithm 1.

Algorithm 1 Generation of $R_T^\star$

Initialize $R_T^\star = 1_{p\times p}$
for each $i = 1, \ldots, m$, $k = 1, \ldots, p$ **do**
    if $T_{ik} == 0$ **then**
        for each $j = 1, \ldots, p$ **do**
            if $T_{ij} == 1$ **then**
                $(R_T^\star)_{jk} \leftarrow 0$
            end if
        end for
    end if
end for

The algorithm has polynomial complexity $O(mp^2)$ due to the three nested loops and is thus computationally efficient. The idea behind Algorithm 1 is to only set an entry of $R_T^\star$ to 0 if the condition $TR_T^\star \leq T$ would be violated. We have the following result about $R_T^\star$.

Theorem 2 *Consider a binary matrix $T \in \{0,1\}^{m\times p}$, and define $\mathcal{R}_T := \{R \in \{0,1\}^{p\times p} \mid R \geq I_p, (11) \text{ holds}\}$. Then,*

1. *There exists a unique $R_T^\star \in \mathcal{R}_T$ such that $R_T^\star \geq R^{p-1}, \forall R \in \mathcal{R}_T$.*

2. *Such $R_T^\star$ can be computed via Algorithm 1.*

Proof Let $R_T^\star$ be the unique binary matrix generated by Algorithm 1. It is easy to check that $TR_T^\star \leq T$ by construction. It follows that $T(R_T^\star)^{p-1} \leq \cdots \leq TR_T^\star \leq T$, so $R_T^\star \in \mathcal{R}_T$.

Next, consider any binary matrix $R \in \mathcal{R}_T$. By definition, we have that $TR^{p-1} \leq T$ and so $(R^{p-1})_{jk} = 0$ whenever $T_{ij} = 1$ and $T_{ik} = 0$. Then, $R^{p-1} \leq R_T^\star$ since $(R_T^\star)_{jk}$ is set to 0 by Algorithm 1 if and only if $T_{ik} = 0$ and $T_{ij} = 1$. Therefore, we have $R^{p-1} \leq R_T^\star$, $\forall R \in \mathcal{R}_T$.

The relevance of the above result to distributed control is stated in the following Corollary.

Corollary 2 *Given a binary matrix $T \leq S$ compute $R_T^\star$ as per Algorithm 1. Then, for every fixed invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$, $\mathcal{P}_{T,R_T^\star}$ is the tightest convex restriction of $\mathcal{P}_K$ among those in the form $\mathcal{P}_{T,R^{p-1}}$ with $R \in \mathcal{R}_T$.*

Proof Fix an invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$ and consider the problems $\mathcal{P}_{T,R^{p-1}}$ and $\mathcal{P}_{T,R_T^\star}$, where $R \in \mathcal{R}_T$ and $R_T^\star$ is generated by Algorithm 1. By Theorem 2, we have $R^{p-1} \leq R_T^\star$, meaning that $\text{Sparse}(R^{p-1}) \subset \text{Sparse}(R_T^\star)$.

The only difference between problem $\mathcal{P}_{T,R^{p-1}}$ and problem $\mathcal{P}_{T,R_T^\star}$ is: $\mathcal{P}_{T,R^{p-1}}$ requires $\mathbf{X}_Q\mathbf{\Gamma} \in \text{Sparse}(R^{p-1})$ while $\mathcal{P}_{T,R_T^\star}$ requires $\mathbf{X}_Q\mathbf{\Gamma} \in \text{Sparse}(R_T^\star)$. Therefore, we conclude that $\mathcal{P}_{T,R_T^\star}$ admits the largest feasible region among all $\mathcal{P}_{T,R^{p-1}}$ with $R \in \mathcal{R}_T$. This completes our proof.

Given a fixed invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$ and a binary matrix $T \leq S$, we have provided an efficient procedure to select a tight convex restriction for $\mathcal{P}_K$. However, optimally choosing $\mathbf{\Gamma}$ and T is also a non-trivial task which we leave for future work. We remark that in the lack of any further insight, one can always choose $T = S$ and $\mathbf{\Gamma} = I_p$ and still obtain distributed controllers with tight sub-optimality gaps, as will be shown experimentally in Section 5. Furthermore, as shown in Section 4, the trivial choice $T = S$ and $\mathbf{\Gamma} = I_p$ combined with Algorithm 1 for choosing R is sufficient to recover and extend the optimality results of [7], [17] which are based on the Quadratic Invariance (QI) notion. We conclude this section by providing an example to illustrate the SI approach.

Example 1 Motivated by the numerical example in [7], let us consider the unstable plant

$$\mathbf{G} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & 0 & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & \frac{1}{s-1} & \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s-1} \end{bmatrix},$$

with

$$\mathbf{P}_{11} = \begin{bmatrix} \mathbf{G} & 0_{5\times5} \\ 0_{5\times5} & 0_{5\times5} \end{bmatrix}, \quad \mathbf{P}_{12} = \begin{bmatrix} \mathbf{G} \\ I_5 \end{bmatrix}, \quad \mathbf{P}_{21} = \begin{bmatrix} \mathbf{G} & I_5 \end{bmatrix}.$$

Our goal is to design a stabilizing controller $\mathbf{K}$ which minimizes $\|f(\mathbf{K})\|_{\mathcal{H}_2}$ and satisfies the sparsity pattern below:

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

This information structure is depicted in Figure 2.

Here, we apply the proposed SI approach and Algorithm 1 for sparsity design in order to obtain a convex restriction of $\mathcal{P}_K$. For this instance, we choose to fix $T = S$ and $\mathbf{\Gamma} = I_p$. According to Theorem 2 and Corollary 2, the tightest convex restriction of $\mathcal{P}_K$ such that $TR^{p-1} = SR^{p-1} \leq S$ is $\mathcal{P}_{S,R_S^\star}$, where $R_S^\star$

$$R_S^\star = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

is generated via Algorithm 1. Given a doubly coprime factorization of $\mathbf{G}$, any solution of $\mathcal{P}_{S,R_S^\star}$ is in the form $\mathbf{K} = \mathbf{Y}_Q(\mathbf{X}_Q)^{-1} \in \mathcal{C}_{\text{stab}} \cap \text{Sparse}(S)$, where $\mathbf{Y}_Q \in \text{Sparse}(T)$, $\mathbf{X}_Q \in \text{Sparse}(R_S^\star)$ and $(\mathbf{X}_Q)^{-1} \in \text{Sparse}(R_S^\star)$.

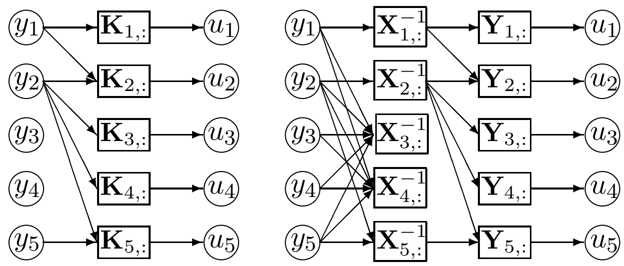

Figure 2: In the figure, we denote as $\mathbf{K}_{i,:}$, $\mathbf{Y}_{j,:}$, $\mathbf{X}^{-1}_{k,:}$ the ith, jth and kth row of $\mathbf{K}$, $\mathbf{Y}_Q$ and $\mathbf{X}_Q^{-1}$ respectively. For every non-zero entry of $\mathbf{K}_{i,:}$, $\mathbf{Y}_{j,:}$ or $\mathbf{X}^{-1}_{k,:}$ the corresponding signal enters the block with an arrow, thus representing the information flow from measured outputs to control signals. The scheme on the left represents the desired sparsity pattern S for controller $\mathbf{K}$. The scheme on the right represents the sparsity pattern of controllers that are feasible for $\mathcal{P}_{S,R_S^\star}$, i.e. those in the form $\mathbf{Y}_Q(\mathbf{X}_Q)^{-1}$ with $\mathbf{Y}_Q \in \text{Sparse}(S)$ and $\mathbf{X}_Q \in \text{Sparse}(R_S^\star)$.

Remark 3 (*Performance improvement*) Note that allowing off-diagonal entries of $\mathbf{X}_Q$ to be non-zero through the optimized choice of $R_S^\star$ is beneficial for performance improvement. Indeed, $\mathbf{X}_Q = I_p + \mathbf{G}\mathbf{Y}_Q$ by (8) and $\mathbf{Y}_Q \in \text{Sparse}(S)$, implies that $(\mathbf{G}\mathbf{Y}_Q)_{3,2}$ and $(\mathbf{G}\mathbf{Y}_Q)_{4,2}$ can be non-zero. By letting $\mathbf{X}_Q \in \text{Sparse}(R_S^\star)$ we thus allow for more freedom and to a larger feasible region in optimization compared to the immediate idea $\mathbf{X}_Q \in \text{Sparse}(I_5)$ as per [14, 15, 21]. This additional freedom can be seen graphically on the right side of Figure 2; the information flow from outputs to control inputs remains the same as the one encoded by S, but we allow for as many arrows as possible in the first stage from outputs to the rows of $\mathbf{X}^{-1}$, thus maximizing the degrees of freedom in the optimization. In Section 5 we will numerically solve $\mathcal{P}_{S,R_S^\star}$ for this example and show that performance improvement over the method of [17] is obtained.

4 Beyond Quadratic Invariance

We start by recalling the well-known notion of Quadratic Invariance (QI) [7] in Subsection 4.1, and its application to the design of globally optimal [7] and sub-optimal [17] distributed dynamic output-feedback controllers in Subsection 4.2. In Subsections 4.3, 4.4 we show that the suggested SI notion strictly goes beyond that of QI: 1) the controllers obtained within the SI framework perform at least as well as those obtained by [7] and [17]; 2) we show through examples that using the SI notion we can recover globally optimal controllers even when QI does not hold, and that strict performance improvements over [17] can be obtained in general. Last, in Subsection 4.5 we discuss applicability of SI to computing distributed static controllers, whereas the QI notion is not applicable.

4.1 Quadratic Invariance

The celebrated work of [7] characterized conditions on $\mathbf{G}$ and $\text{Sparse}(S)$ under which $\mathcal{P}_K$ admits an exact convex reformulation in the Youla parameter $\mathbf{Q}$, denoted as *quadratic invariance* (QI).

Definition 2 (Quadratic invariance [7]) *A subspace $\mathcal{K} \subseteq \mathcal{R}_p^{m\times p}$ is QI with respect to $\mathbf{G}$ if*

$$\mathbf{K}\mathbf{G}\mathbf{K} \in \mathcal{K}\,, \quad \forall \mathbf{K} \in \mathcal{K}\,.$$

It is shown that given a controller $\mathbf{K}_{\text{nom}} \in \text{Sparse}(S)$ that stabilizes $\mathbf{G}$ and is itself stable, there exists a parametrization such that $\mathbf{K} \in \text{Sparse}(S) \Leftrightarrow \mathbf{Q} \in \text{Sparse}(S)$ [7]. Accordingly, a convex optimization problem equivalent to $\mathcal{P}_K$ is obtained. The requirement of a stable and stabilizing controller $\mathbf{K}_{\text{nom}}$ was removed in [22]. One main result from [22] is as follows:

Theorem 3 (Theorem IV.2 of [22]) *Consider any doubly-coprime factorization of* $\mathbf{G}$ *and let* $\text{Sparse}(S)$ *be QI with respect to* $\mathbf{G}$*. Then, the following two statements hold:*

1. *If* $\mathbf{Q} \in \mathcal{RH}_\infty^{m \times p}$ *is such that* $\mathbf{Y}_Q \in \text{Sparse}(S)$*, then* $\mathbf{K} = \mathbf{Y}_Q \mathbf{X}_Q^{-1}$ *is a stabilizing controller in* $\text{Sparse}(S)$*.*

2. *For any* $\mathbf{K} \in \mathcal{C}_{\text{stab}} \cap \text{Sparse}(S)$ *there exists* $\mathbf{Q} \in \mathcal{RH}_\infty^{m \times p}$ *for which* $\mathbf{Y}_Q \in \text{Sparse}(S)$ *and* $\mathbf{K} = \mathbf{Y}_Q \mathbf{X}_Q^{-1}$*.*

According to Theorem 3, if $\text{Sparse}(S)$ is QI with respect to $\mathbf{G}$, then $\mathcal{P}_K$ can be equivalently reformulated as

$$\begin{aligned} \underset{\mathbf{Q} \in \mathcal{RH}_\infty^{m \times p}}{\text{minimize}} \quad & \|\mathbf{T}_1 - \mathbf{T}_2 \mathbf{Q} \mathbf{T}_3\| \\ \text{subject to} \quad & (5),\ (6),\ \mathbf{Y}_Q \in \text{Sparse}(S)\,. \end{aligned} \tag{12}$$

The optimal solution $\mathbf{Q}^\star$ of (12) can be used to recover the *globally* optimal solution $\mathbf{K}^\star$ of $\mathcal{P}_K$ via $\mathbf{K}^\star = \mathbf{Y}_{Q^\star} \mathbf{X}_{Q^\star}^{-1}$.

4.2 Convex restrictions for non-QI information structures

When $\text{Sparse}(S)$ is not QI with respect to $\mathbf{G}$, the authors of [17] proposed finding a binary matrix $T_{\text{QI}} < S$ such that $\text{Sparse}(T_{\text{QI}})$ is QI with respect to $\mathbf{G}$. Then, the constraint $\mathbf{Y}_Q \mathbf{X}_Q^{-1} \in \text{Sparse}(S)$ of problem $\mathcal{P}_Q$ can be replaced by $\mathbf{Y}_Q \in \text{Sparse}(T_{\text{QI}})$, and any feasible $\mathbf{Q}$ for this convex program will correspond to a feasible controller

$$\begin{aligned} \mathbf{K} = \mathbf{Y}_Q \mathbf{X}_Q^{-1} &\in \mathcal{C}_{\text{stab}} \cap \text{Sparse}(T_{\text{QI}}) \\ &\subseteq \mathcal{C}_{\text{stab}} \cap \text{Sparse}(S)\,. \end{aligned} \tag{13}$$

This inclusion (13) directly follows from Theorem 3 and the fact that $\text{Sparse}(T_{\text{QI}}) \subset \text{Sparse}(S)$.

A challenge of this approach is to compute T_{QI} such that $\text{Sparse}(T_{\text{QI}})$ is QI and as close as possible to S in order to reduce conservatism, in the sense that $\|S\|_0 - \|T_{\text{QI}}\|_0$ is minimized. In general, there might be multiple choices of T_{QI} with the same cardinality. Furthermore, the QI condition $T_{\text{QI}} \Delta T_{\text{QI}} \leq T_{\text{QI}}$ of [7, Theorem 26], where $\Delta = \text{Struct}(\mathbf{G})$, is nonlinear in T_{QI}. For these reasons, a procedure to compute a closest QI subset of S in polynomial time was not provided in [17]. Instead, we have shown that the polynomial time Algorithm 1 can be used within the SI framework to find a convex restriction for any given $T \leq S$. In the next subsections, we show that the recovered controllers perform at least as well as those based on the notion of QI by choosing $T \leq S$ appropriately, and can be strictly more performing in general even with the trivial choice $T = S$.

4.3 Connections of SI with QI

Here, we show that it is not necessary to check the QI property in order to obtain a globally optimal solution. Note that checking the property of QI before solving $\mathcal{P}_K$ was proposed in [7] and required in many subsequent work. Indeed, the approach in [7] is guaranteed to yield feasible solutions for $\mathcal{P}_K$ only if QI holds. Instead, our technique can be directly applied given S without first checking QI. This result is summarized in the following theorem and corollary.

Theorem 4 *Let* $\Delta = \text{Struct}(\mathbf{G})$ *and let* $R_S^\star$ *be the binary matrix generated by Algorithm 1 with* $T = S$*. The following statements are equivalent.*

i) $\text{Sparse}(S)$ *is QI with respect to* $\mathbf{G}$*.*

ii) $R_S^\star \geq I_p + \Delta S$*, where* $R_S^\star$ *is generated by Algorithm 1 with* $T = S$*.*

Proof i) $\Rightarrow$ ii): Suppose that $\text{Sparse}(S)$ is QI with respect to $\mathbf{G}$. We have that $S\Delta S \leq S$ by [7, Theorem 26], implying that $S(I_p + \Delta S) \leq S$ and ultimately $S(I_p + \Delta S)^{p-1} \leq S$. We have that $R_S^\star \geq I_p$ and $SR_S^\star \leq S$ by construction. It follows that $S(R_S^\star)^{p-1} \leq \ldots \leq SR_S^\star \leq S$. Also, according to Theorem 2, we have $R_S^\star \geq R$, $\forall R \geq I_p$ such that $SR^{p-1} \leq S$. By posing $R = I_p + \Delta S$, we have shown above that $SR^{p-1} \leq S$. Hence, $R_S^\star \geq R = (I_p + \Delta S)$.

ii) $\Rightarrow$ i): Suppose that $R_S^\star \geq I_p + \Delta S$, which implies $(R_S^\star)^{p-1} \geq (I_p + \Delta S)^{p-1}$. By definition of $R_S^\star$, we have observed that $S(R_S^\star)^{p-1} \leq S$. It follows that

$$S(I_p + \Delta S)^{p-1} \leq S(R_S^\star)^{p-1} \leq S\,. \tag{14}$$

Combining (14) with the fact that $(I_p + \Delta S) \geq I_p$, we have

$$S(I_p + \Delta S) \leq S(I_p + \Delta S)^{p-1} \leq S\,.$$

This implies $S\Delta S \leq S$ which is equivalent to QI by [7, Theorem 26].

Corollary 3 *The following statements are equivalent.*

i) $\text{Sparse}(S)$ *is QI with respect to* $\mathbf{G}$*.*

ii) $\mathcal{P}_K$ *is equivalent to* $\mathcal{P}_{S,R_S^\star}$ *with* $\mathbf{\Gamma} = I_p$*, where* $R_S^\star$ *is the binary matrix generated by Algorithm 1 with* $T = S$*.*

Proof It is well-known [22] that (12) is equivalent to $\mathcal{P}_K$ if and only QI holds. It remains to show that $\mathcal{P}_{S,R_S^\star}$ is equivalent to (12) if and only if QI holds.

We first show that $\mathbf{X}_Q$ lies in $\text{Sparse}(I_p+\Delta S)$ for every $\mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}$ such that $\mathbf{Y}_Q \in \text{Sparse}(S)$. Indeed, by (8) we have $\mathbf{X}_Q = I_p + \mathbf{G}\mathbf{Y}_Q$ for every $\mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}$ and thus $\mathbf{X}_Q \in \text{Sparse}(I_p + \Delta S)$. We have shown in Theorem 4 that QI is equivalent to $R_S^\star \geq I_p + \Delta S$, where $R_S^\star$ is generated by Algorithm 1. It follows that the constraint $\mathbf{Y}_Q\mathbf{\Gamma} = \mathbf{Y}_Q \in \text{Sparse}(S)$ makes the constraint $\mathbf{X}_Q\mathbf{\Gamma} = \mathbf{X}_Q \in \text{Sparse}(R_S^\star)$ redundant and thus $\mathcal{P}_{S,R_S^\star}$ with $\mathbf{\Gamma} = I_p$ is equivalent to (12). This concludes the proof.

Essentially, Theorem 4 shows that QI is equivalent to $R_S^\star \geq I_p+\Delta S$. Since $\mathbf{X}_Q \in \text{Sparse}(I_p+\Delta S)$ by (8) when $\mathbf{Y}_Q \in \text{Sparse}(S)$, the constraint $\mathbf{X}_Q \in \text{Sparse}(R_S^\star)$ becomes redundant if and only if QI holds and the convex program we obtain with SI, namely $\mathcal{P}_{S,R_S^\star}$ with $\mathbf{\Gamma} = I_p$, is equivalent to $\mathcal{P}_K$ due to the results of [7].

Theorems 1, 2 and 4, and Corollaries 1–3 can be summarized as follows. Given any distributed control problem $\mathcal{P}_K$, one can always cast and solve its convex restriction $\mathcal{P}_{S,R_S^\star}$, where $R_S^\star$ is generated by Algorithm 1. If $\mathcal{P}_{S,R_S^\star}$ is feasible, its optimal solution is also feasible for $\mathcal{P}_K$, and is certified to be globally optimal if $\text{Sparse}(S)$ is QI with respect to $\mathbf{G}$. We remark that verifying QI is optional and can be done a-posteriori to check global optimality of the solution, but QI is not part of the controller design procedure in the SI framework. Hence, Theorem 4 expands the applicability of convex programming to compute distributed controllers for arbitrary systems and information structures, while maintaining previous global optimality results.

Example 2 Consider the unstable system and the sparsity pattern S of Example 1. We can verify that $S\Delta S \not\leq S$, where $\Delta = \text{Sparse}(\mathbf{G})$, and hence Sparse($S$) is not QI with respect to $\mathbf{G}$. Instead, let us consider the new sparsity pattern

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} . \tag{15}$$

We can verify that $S_2\Delta S_2 \leq S_2$. Hence, Sparse(S_2) is QI with respect to $\mathbf{G}$. By applying Algorithm 1 we obtain

$$R^\star_{S_2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad I_p + \Delta S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$R^\star_{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad I_p + \Delta S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

In accordance with Theorem 4 we have that $R^\star_{S_2} \geq I_p + \Delta S_2$, but $R^\star_S \not\geq I_p + \Delta S$ (see the entries highlighted in red). By Corollary 3, we conclude that the convex program $\mathcal{P}_{S_2, R^\star_{S_2}}$ with $\mathbf{\Gamma} = I_p$ is equivalent to $\mathcal{P}_K$, while $\mathcal{P}_{S,R^\star_S}$ is a convex restriction of $\mathcal{P}_K$ for every invertible $\mathbf{\Gamma} \in \mathcal{R}_p^{p\times p}$.

Next, we show that SI generalizes the class of restrictions of [17], based on finding QI subsets of Sparse(S) which are nearest to Sparse(S). The result is a straightforward corollary of Theorem 4.

Corollary 4 *Let Sparse(T_{QI}) $\subseteq$ Sparse(S) be QI with respect to* $\mathbf{G}$ *and let* $\|S\|_0 - \|T_{QI}\|_0$ *be minimal as proposed in [17]. Then, there exists* $T \leq S$ *such that* $J^\star \leq J_{QI}$*, where* $J^\star$ *is the minimum cost of* $\mathcal{P}_{T,R^\star_T}$ *with* $\mathbf{\Gamma} = I_p$*, and* J_{QI} *is the minimum cost of problem (12) with the constraint* $\mathbf{Y}_Q \in \text{Sparse}(S)$ *replaced by* $\mathbf{Y}_Q \in \text{Sparse}(T_{QI})$*.*

Proof Let $T = T_{\text{QI}}$. Since Sparse(T_{QI}) is QI with respect to $\mathbf{G}$, we have $R^\star_T \geq I_p + \Delta T$ by Theorem 4. Hence, for every $\mathbf{Y}_Q\mathbf{\Gamma} = \mathbf{Y}_Q \in \text{Sparse}(T)$, the matrix $\mathbf{X}_Q = I_p + \mathbf{G}\mathbf{Y}_Q$ belongs to Sparse($I_p + \Delta T$) for every $\mathbf{Q} \in \mathcal{R}_\infty^{m\times p}$ and the constraint $\mathbf{X}_Q\mathbf{\Gamma} = \mathbf{X}_Q \in \text{Sparse}(R^\star_T)$ is redundant. It follows that the choice $T = T_{\text{QI}}$ achieves $J^\star = J_{\text{QI}}$. Therefore, there exists a choice of T such that the optimal solution of $\mathcal{P}_{T,R^\star_T}$ with $\mathbf{\Gamma} = I_p$ performs at least as well as that of the problem obtained by considering a nearest QI subset as suggested in [17]. This completes our proof.

Corollary 4 proves that the class of convex restrictions considered in [17] is a special case in the framework of SI, obtained by choosing $T = T_{\text{QI}}$ and computing $R^\star_{T_{\text{QI}}}$ with our Algorithm 1. Furthermore, it is possible to choose $T \leq S$ to obtain strictly more performing convex restrictions, as we will show numerically in Section 5.

4.4 Strictly Beyond QI

So far, we have shown that the SI framework naturally recovers the previous QI results of [7] and [17] as specific cases by using Algorithm 1. Here and in Section 5, we show through examples the stronger results that

1. SI can recover globally optimal solutions when QI does *not* hold,
2. strictly better performance than the approach of [17] can be obtained.

For point 2), we refer to the numerical results in Section 5. For point 1), we consider an example taken from [12].

Example 3 Consider the optimal control problem:

$$\begin{aligned} \underset{\mathbf{K}(z)}{\text{minimize}} \ & \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \mathbb{E}||x(t)||_2^2 \\ \text{subject to } & x(t+1) = Ax(t) + u(t) + w(t)\,, \\ & u(z) = \mathbf{K}(z)x(z)\,, \quad \mathbf{K}(z) \in \text{Sparse}(A^{\text{bin}})\,, \end{aligned}$$

where $z \in e^{j\mathbb{R}}$, $A \in \mathbb{R}^{n\times n}$, $A^{\text{bin}} = \text{Struct}(A)$ and $w(t)$ denotes i.i.d. disturbances distributed according to a normal distribution $\mathcal{N}(0_{n\times 1}, I_n)$. The discrete-time transfer function of this system is $\mathbf{G}(z) = (zI_p - A)^{-1}$. This problem without the sparsity constraint on $\mathbf{K}$ is known as the LQR problem. By adding the sparsity constraint, it is an instance of $\mathcal{P}_K$ in discrete-time. Notice that QI does not hold whenever the graph defined by A is strongly connected because $\Delta = \text{Struct}(\mathbf{G}(z)) = \text{Struct}\left((zI_n - A)^{-1}\right)$ is equal to $1_{n\times n}$ in general, and so $A^{\text{bin}}\Delta A^{\text{bin}} \not\leq A^{\text{bin}}$ thus violating QI.

The reason to consider a discrete-time instance of $\mathcal{P}_K$ is that its optimal solution can be computed by solving the corresponding discrete-time Riccati equation [23] analytically. Indeed, we verify that the globally optimal solution is $\mathbf{K}(z) = -A$. Now, consider problem $\mathcal{P}_{T,R}$ with $\mathbf{\Gamma}(z) = \mathbf{G}(z)$, $T = A^{\text{bin}}$ and $R = R^\star_{A^{\text{bin}}}$. We can verify that a feasible solution for $\mathcal{P}_{T,R}$ is $\mathbf{Y}_Q(z) = -\frac{A}{z}(zI_n - A)$, because

$$\mathbf{Y}_Q\mathbf{\Gamma} = \mathbf{Y}_Q(zI_n - A)^{-1} = -\frac{A}{z} \in \text{Sparse}(A^{\text{bin}})\,.$$

This implies $\mathbf{X}_Q(z) = I_n - \frac{A}{z}$ by (8). Hence, $\mathbf{X}_Q(z)\mathbf{\Gamma}(z) = \mathbf{X}_{\mathbf{Q}}(z)(zI_n - A)^{-1} = \frac{I_n}{z}$. Since $R^\star_{A^{\text{bin}}} \geq I_n$ by design (see Algorithm 1), we have $\mathbf{X}_Q(z)\mathbf{\Gamma}(z) \in \text{Sparse}(R^\star_{A^{\text{bin}}})$ as desired. It is immediate to verify that the resulting controller is $\mathbf{K}(z) = \mathbf{Y}_{\mathbf{Q}}(z)\mathbf{X}_Q(z)^{-1} = -A$. We conclude that, despite a lack of QI, a convex approximation which contains the global optimum of $\mathcal{P}_K$ is found by using the proposed SI approach.

Remark 4 The global optimality result for this example was also obtained using the system level parametrization in [12]. The sparsities for the system level parameters in [12] were chosen empirically, while we provide an explicit methodology based on the SI condition (10) and Algorithm 1. Furthermore, we wish to clarify that obtaining global optimality certificates for $\mathcal{P}_K$ for systems with non-QI constraints is still an open problem, which is not addressed neither by the system level approach [12] nor by our SI framework. Both our approach and that of [12] can certify optimality of the solution because the optimal solution of this simple instance is already known analytically.

4.5 SI for static controller design

We conclude this section by highlighting another advantage of the SI notion over the QI notion; SI can be used to compute distributed static control policies in a convex way, that is policies in the form $u(t) = Ky(t)$ where K is a *real* matrix in Sparse(S). This topic has been thoroughly studied in our earlier work [16], where we derived a primitive version of the SI notion only applicable to the static controller case. Here, we highlight that in contrast to the QI notion, SI can be applied both for static and dynamic distributed control design.

The main observation is that the Youla parametrization cannot achieve a convexification of the static controller design problem in general, because the constraint $\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \in \mathbb{R}^{m\times p}$ is non-convex in $\mathbf{Q}$. Consequently, a different parametrization must be used and the QI property, tightly linked to the use of a Youla-like parametrization, will not be relevant anymore. The most well-known techniques to convexify the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm-optimal state-feedback static controller design problems are based on computing appropriate quadratic Lyapunov functions through Linear Matrix Inequalities (LMI); see [24, 25] for a comprehensive review. The more general case of static output-feedback is known to be NP-hard [5] and an exact convex formulation does not exist.

As we illustrated in [16], when the distributed static control problem is formulated through LMIs, the controller is recovered as $K = YX^{-1}$, where Y and X are real decision variables, X is symmetric positive semidefinite and $V(x) = x^\mathsf{T}X^{-1}x$ is a quadratic Lyapunov function for the closed-loop system. If the controller must lie in a sparsity subspace Sparse(S), the only source of non-convexity stems from requiring that $YX^{-1} \in$ Sparse(S). This expression for the static controller in terms of the decision variables matches that of $\mathbf{K} = \mathbf{Y}_Q\mathbf{X}_Q^{-1}$, which is valid for dynamic controllers in terms of the Youla parameter. According to Theorem 1 and Corollary 1, convex restrictions can be obtained by choosing binary matrices T and R as per (10) that satisfy the SI condition (9), and requiring that $Y\Gamma \in$ Sparse(T) and $X\Gamma \in$ Sparse(R) for any invertible real matrix $\Gamma \in \mathbb{R}^{n\times n}$. We refer the interested reader to [16] for details.

5 Numerical Results

With the goal of providing insight into our proposed method and showing its potential benefits, we continue here our Example 1, and present numerical results.

Example 1 (continued) Consider the optimal distributed controller design problem formulated in Example 1. We have observed in Example 2 that Sparse(S) is not QI with respect to $\mathbf{G}$. As we have summarized in Section 4.2, [17] suggests identifying a binary matrix $T_{\text{QI}} < S$ such that Sparse(T_{QI}) is QI with respect to $\mathbf{G}$ ans $\|S\|_0 - \|T_{\text{QI}}\|_0$ is minimized. In this case, we verify by inspection that S_2 in (15) is the only QI sparsity pattern T_{QI} such that $\|S\|_0 - \|T_{\text{QI}}\|_0 \leq 2$. As suggested in [17], we can thus substitute the constraint $\mathbf{Y}_Q(\mathbf{X}_Q)^{-1} \in$ Sparse(S) with $\mathbf{Y}_Q \in$ Sparse(S_2) and the corresponding convex program is a restriction of $\mathcal{P}_K$. Our goal is to compare the minimal cost of this convex restriction and that of $\mathcal{P}_{S,R_S^\star}$ with $\mathbf{\Gamma} = I_p$ obtained through SI.

Finite-dimensional approximation: Since the convex programs we have cast are infinite-dimensional, due to the decision variables being transfer matrices whose order is not fixed, it is necessary to resort to finite-dimensional approximation techniques. Here, we adapt the semidefinite programming technique of [26] to the continuous-time case and to the $\mathcal{H}_2$ norm, by exploiting standard results from [25, 27]. It is beyond the scope of this paper to compare this methodology with different finite-dimensional approximation techniques available in the literature. The key idea behind the

Table 1: Numerical results for Example 1 using the nearest QI subset approach [17] and the proposed SI approach $\mathcal{P}_{S,R_S^\star}$. N is the order of finite approximation, and the lowest cost for each N is marked by *.

	Nearest QI subset	$\mathcal{P}_{S,R_S^\star}$
$N=1$	10.5885*	10.5885*
$N=2$	8.4031	8.3859*
$N=3$	8.2932	8.2689*
$N=4$	8.2056	8.1814*
$N=5$	8.1986	8.1748*
$N=6$	8.1972	8.1736*

approach of [26] is summarized as follows. Consider the set

$$\left\{(s+a)^{-k}\right\}_{k=0}^{N}, \tag{16}$$

where $N \in \mathbb{N}$ and $a > 0$ is any real number. By [28], for every $\mathbf{g}' \in \mathcal{RH}_\infty$ in continuous-time there exists $\mathbf{g}$ in the subspace spanned by (16) with $N \to \infty$ such that $\|\mathbf{g}-\mathbf{g}'\|<\epsilon$ for every $\epsilon > 0$, where $\|\cdot\|$ can be, for instance, the $\mathcal{H}_2$, $\mathcal{H}_\infty$ or $\mathcal{L}_1$ norm [28]. Hence, optimizing over the subspace spanned by (16) for $N \to \infty$ yields the same results as optimizing over $\mathcal{RH}_\infty$. Now if the Youla parameter $\mathbf{Q}$ is parametrized as

$$\mathbf{Q} = \sum_{i=0}^{N} Q[i](s+a)^{-i}, \tag{17}$$

for some $N \in \mathbb{N}$, and the real matrices $Q[i]$ for all i are decision variables, we have that $\mathbf{Q} \in \mathcal{RH}_\infty^{m\times p}$ and a finite-dimensional approximation of our convex program is obtained. The corresponding $\mathcal{H}_2$ norm of $f(\mathbf{K})$ can be then encoded through semidefinite constraints as per the results of [25–27].

Numerical results: As outlined above, we solved finite-dimensional approximations of the convex restriction proposed in [17] and of our convex restriction $\mathcal{P}_{S,R_S^\star}$ with $\mathbf{\Gamma} = I_p$ obtained through SI. The doubly-coprime factorization for $\mathbf{G}$ is computed as per [7, Theorem 17] using the stable and stabilizing controller $\mathbf{K}_{\text{nom}}$ suggested in [7, Page 1995]. In (17), we chose $a = 2$, as it was found to yield the lowest cost, and increased values of N until the improvement on the cost was negligible. The semidefinite programs were solved with MOSEK [29], called through MATLAB via YALMIP [30], on a standard laptop computer.

As listed in Table 1, our SI method leads to a lower cost for all N compared with the nearest QI subset approach. This improvement is possible because the entries $(1,1)$ and $(2,1)$ are allowed to be non-zero in the controllers corresponding to solutions of $\mathcal{P}_{S,R_S^\star}$, whereas they are forced to be 0 to comply with the sparsity of the nearest QI subset S_2 as per [17]. For instance, let $\mathbf{K}^\star$ be the controller recovered by the solution of the finite-dimensional approximation of $\mathcal{P}_{S,R_S^\star}$ with $a = 2$ and $N = 2$, where $N = 2 < 6$ is chosen in the interest of a shorter expression for $\mathbf{K}^\star_{1,1}(s)$, which is reported in the footnote[2]. Since $\mathbf{K}^\star_{1,1}(s) \neq 0$, this controller $\mathbf{K}^\star(s)$ with lower cost could not be computed with the method of [17], according to which $\mathbf{K}^\star_{1,1}(s)$ is forced to be zero.

For completeness, we additionally considered the nearest QI superset of S defined as the binary matrix $S_3 \geq S$ such that S_3 is QI and $\|S_3\|_0 - \|S\|_0$ is minimized [17]. The QI superset is unique

[2] $\mathbf{K}^\star_{1,1}(s) = -\frac{5.3138+46.445s+189.96s^2+484.78s^3+867.85s^4+1160.5s^5+1203.1s^6+989.09s^7+652.73s^8+347.26s^9+148.47s^{10}+50.451s^{11}+13.341s^{12}+2.6488s^{13}+0.3714s^{14}+0.033s^{15}+0.0014s^{16}}{20.455+184.1s+778.84s^2+2064.8s^3+3857.2s^4+5406.4s^5+5904.8s^6+5144.1s^7+3623.8s^8+2077.5s^9+969.37s^{10}+365.84s^{11}+110.18s^{12}+25.881s^{13}+4.5687s^{14}+0.5697s^{15}+0.0447s^{16}+0.0017s^{17}}$.

and is computed with the algorithm (13)-(14) of [17]:

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} .$$

By setting $a = 2$ and $N = 6$, the corresponding globally optimal controller $\mathbf{K}_3^\star \in \text{Sparse}(S_3)$ yields a cost of 8.0833. Since $\mathbf{K}_3^\star$ does not lie in Sparse(S), and Sparse(S_3) is QI with respect to $\mathbf{G}$, the value 8.0833 serves as a lower-bound on the optimal cost of $\mathcal{P}_K$. We conclude that, for $a = 2$ and $N = 6$, our SI solution improves on that of [17] based on QI subsets by at least $\frac{8.1972-8.1736}{8.1972-8.0833} = 20.7\%$.

6 Conclusions

We have proposed the framework of Sparsity Invariance (SI) for convex design of optimal and near-optimal distributed controllers. One main insight is that the proposed SI framework offers a direct generalization of previous design methods based on the notion of Quadratic Invariance (QI). Indeed, SI can be directly applied to any systems and information structures. The recovered solution is globally optimal when QI holds and performs at least as well as the nearest QI subset when QI does not hold. We have shown the potential benefits of SI over previous methods through examples, and remarked that SI is naturally applicable to distributed static controller design.

Since the condition (10) is necessary and sufficient for the SI property (9), our results approach the limits in performance of convex restrictions of the distributed control problem which are based on structural conditions for the Youla parameter. This opens up the question of whether different and more performing design methodologies can be developed for this challenging problem. Another direction for research is to further refine the SI approach, by developing tractable heuristics to optimally design the binary matrices T and R and the parameter $\mathbf{\Gamma}$ simultaneously based on the knowledge of the system $\mathbf{P}$. This could potentially improve upon Algorithm 1. Finally, we note that similar to QI, SI is an algebraic condition independent of the chosen parameterization of stabilizing controllers. In this work, we solely focus on the Youla parameterization. A detailed comparison when combining SI with the input-output [31] and the system-level [12] parametrizations is left for future work.

Appendix

6.1 Proof of Theorem 1

The proof relies on two Lemmas. We report the proof of Lemma A1 in Appendix 6.2 and the proof of Lemma A2 in Appendix 6.3.

Lemma A1 *Let $R \in \{0,1\}^{p\times p}$ with $R \geq I_p$. Then,*

1. *For any invertible transfer matrix $\mathbf{X}$ in* Sparse (R)*, we have*

$$\text{Struct}\left(\mathbf{X}^{-1}\right) \leq R^{p-1} .$$

2. *There exists an invertible transfer matrix $\mathbf{X} \in \text{Sparse}(R)$ such that*

$$\text{Struct}\left(\mathbf{X}^{-1}\right) = R^{p-1} .$$

Lemma A2 *Let* $T \in \{0,1\}^{m\times p}$ *and* $R \in \{0,1\}^{p\times p}$, *and* $\text{Struct}(\mathbf{W}) = R$. *Then, there exists* $\mathbf{Z} \in \text{Sparse}(T)$ *such that*

$$\text{Struct}(\mathbf{ZW}) = TR\,.$$

We are now ready to prove Theorem 1.

1) $\Rightarrow$ 2): Let $\mathbf{X} \in \text{Sparse}(R)$ be invertible. By Lemma A1 we know that $\mathbf{X}^{-1} \in \text{Sparse}(R^{p-1})$. Now let $\mathbf{Y} \in \text{Sparse}(T)$. Since $TR^{p-1} \leq S$, we have $\mathbf{YX}^{-1} \in \text{Sparse}(S)$.

2) $\Rightarrow$ 1): We prove by contrapositive. First, suppose that $TR^{p-1} \not\leq S$. By the second statement of Lemma A1 it is possible to select $\mathbf{X} \in \text{Sparse}(R)$ such that $\text{Struct}(\mathbf{X}^{-1}) = R^{p-1}$. By the latter and Lemma A2, we can select $\mathbf{Y} \in \text{Sparse}(T)$ such that $\text{Struct}\left(\mathbf{YX}^{-1}\right) = TR^{p-1}$, or equivalently $\mathbf{YX}^{-1} \notin \text{Sparse}(S)$. Next, suppose that $T \not\leq S$. Since $R \geq I_p$ by hypothesis, then $TR \not\leq S$ and $TR^{p-1} \not\leq S$. Hence, the same reasoning applies.

6.2 Proof of Lemma A1

Suppose $\mathbf{X} \in \text{Sparse}(R)$ is invertible. By Cayley-Hamilton's theorem $\sum_{i=0}^{n} \lambda_i \mathbf{X}^i = 0$ where $\{\lambda_i\}_{i=0}^{p}$, $\lambda_i \in \mathcal{R}_p$ for every $i = 1, \ldots, p$ are the coefficients of the characteristic polynomial of $\mathbf{X}$ and $\lambda_0 = \det \mathbf{X} \neq 0$. We remark that Cayley-Hamilton applies as it is valid over square matrices defined over a commutative ring such as the commutative ring of proper transfer functions [32]. By pre-multiplying by $\mathbf{X}^{-1}$ and rearranging the terms we obtain

$$\mathbf{X}^{-1} = -\lambda_0^{-1}(\lambda_1 I_p + \lambda_2 \mathbf{X} + \lambda_3 \mathbf{X}^2 + \cdots + \lambda_p \mathbf{X}^{p-1})\,. \tag{18}$$

Since $R \geq I_p$ we have that $R^a \geq R^b$ for every integer $a \geq b$. Hence, $\lambda_i \mathbf{X}^i \in \text{Sparse}\left(R^{p-1}\right)$ for every i and the first statement follows by (18).

For the second statement, we iteratively construct $\mathbf{X}$ starting from $\mathbf{X} = I_p$. Let $\alpha \in \mathcal{R}_p$. Define $\tilde{\mathbf{X}} = \mathbf{X} + \alpha e_i e_j^{\mathsf{T}}$. Let $\mathbf{X}_{:,i}^{-1} \in \mathcal{R}_p^{p\times 1}$ and $\mathbf{X}_{j,:}^{-1} \in \mathcal{R}_p^{1\times p}$ be the i-th column and the j-th row of $\mathbf{X}^{-1}$ respectively, and let $\mathbf{X}_{ij}^{-1}$ be the entry (i,j) of $\mathbf{X}^{-1}$. Using the Sherman-Morrison identity [33], if $\tilde{\mathbf{X}}$ is invertible we obtain

$$\tilde{\mathbf{X}}_{i,:}^{-1} = \mathbf{X}_{i,:}^{-1} - \frac{\alpha \mathbf{X}_{ii}^{-1}}{1 + \alpha \mathbf{X}_{ji}^{-1}} \mathbf{X}_{j,:}^{-1}\,. \tag{19}$$

Recall that each entry of a transfer matrix is a transfer function defined over $s = j\omega$. Hence, by the definition of an invertible transfer matrix given in Section 2, (19) holds for almost every $\omega \in \mathbb{R}$. From (19), it is easy to verify that, for any i and $\alpha \in \mathcal{R}_p$, if $\mathbf{X}_{ii}^{-1} \neq 0$, then $\tilde{\mathbf{X}}_{ii}^{-1} \neq 0$. It follows that by choosing α such that

$$\begin{aligned}
&\alpha \mathbf{X}_{ji}^{-1} \neq -1 \text{ and } \alpha \left(\mathbf{X}_{ii}^{-1}\mathbf{X}_{jk}^{-1} - \mathbf{X}_{ji}^{-1}\mathbf{X}_{ik}^{-1}\right) \neq \mathbf{X}_{ik}^{-1} \\
&\text{for almost all } \omega \in \mathbb{R}\,, \\
&\forall k \text{ subject to } \mathbf{X}_{jk}^{-1} \text{ and } \mathbf{X}_{ik}^{-1} \text{ are not both null}\,,
\end{aligned} \tag{20}$$

we obtain that

$$\begin{aligned}
&\text{Struct}\left(\tilde{\mathbf{X}}_{i,:}^{-1}\right) = \text{Struct}\left(\mathbf{X}_{i,:}^{-1}\right) + \text{Struct}\left(\mathbf{X}_{j,:}^{-1}\right)\,, \\
&\text{for almost all } \omega \in \mathbb{R}\,.
\end{aligned} \tag{21}$$

The condition (20) is derived by setting the right hand side of (19) to be different from 0 for every k such that $\mathbf{X}_{ik}^{-1}$ and $\mathbf{X}_{jk}^{-1}$ are not both null for every $\omega \in \mathbb{R}$. Observe that α as per (20) always exists, because there is no k such that $\mathbf{X}^{-1}_{ik}$ and $\mathbf{X}^{-1}_{jk}$ are both null for every $\omega \in \mathbb{R}$, and hence $\alpha\left(\mathbf{X}^{-1}_{ii}\mathbf{X}^{-1}_{jk} - \mathbf{X}^{-1}_{ji}\mathbf{X}^{-1}_{ik}\right) \neq \mathbf{X}^{-1}_{ik}$ always admits a solution in $\alpha \in \mathcal{R}_p$. The structural augmentation (21) is exploited in the algorithm below.

Set $\mathbf{X} = I_p$
repeat ▷ max. $(|R| - p)(p-1)$ iterations
  for each (i,j) such that $i \neq j$ and $R_{ij} = 1$ **do**
    Choose α according to (20)
    $\mathbf{X} \leftarrow \mathbf{X} + \alpha e_i e_j^{\mathsf{T}}$
  end for
until Struct$(\mathbf{X}^{-1}) = R^{p-1}$
Return $\mathbf{X}$

The algorithm returns a matrix $\mathbf{X}$ such that Struct$(\mathbf{X}^{-1}) = R^{p-1}$. Specifically, by exploiting (21) we obtain that Struct$(\mathbf{X}^{-1}) \geq R^s$ at the end of the s-th iteration of the "repeat-until" cycle.

6.3 Proof of Lemma A2

Let $\mathbf{Z}$ be any transfer matrix in Sparse(T). Assume that Struct$(\mathbf{ZW}) < TR$. Then, for some (i,j,k) we have that $\mathbf{ZW}_{ij} = 0$ and $T_{ik} = R_{kj} = 1$. We know by hypothesis that $\mathbf{W}_{kj} \neq 0$. Since $\sum_{l=1}^{p} \mathbf{Z}_{il}\mathbf{W}_{lj} = 0$, it is sufficient to update $\mathbf{Z}_{ik}$ with $\mathbf{Z}_{ik} + \alpha$ for any $\alpha \neq 0$ in $\mathcal{R}_p$ to guarantee that $\mathbf{ZW}_{ij} \neq 0$. Furthermore, by choosing $\alpha \neq -\frac{\mathbf{ZW}_{it}}{\mathbf{W}_{kt}}$ for all t such that $\mathbf{ZW}_{it} \neq 0$, we avoid that adding α to $\mathbf{Z}_{ik}$ brings $\mathbf{ZW}_{it}$ to 0 when $\mathbf{ZW}_{it} \neq 0$. Hence, it is always possible to choose k and α such that $\mathbf{ZW} + \alpha e_i e_k^{\mathsf{T}} > \mathbf{ZW}$ and $\mathbf{Z} \in$ Sparse(T). By iterating the procedure for all (i,j) such that Struct$(\mathbf{ZW})_{ij} < TR_{ij}$, we converge to Struct$(\mathbf{ZW}) = TR$.

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