System-level, Input-output and New Parameterizations of Stabilizing Controllers, and Their Numerical Computation

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Abstract

It is known that the set of internally stabilizing controller $C_{\rm stab}$ is non-convex, but it admits convex characterizations using certain closed-loop maps: a classical result is the Youla parameterization, and two recent notions are the system-level parameterization (SLP) and the input-output parameterization (IOP). In this paper, we address the existence of new convex parameterizations and discuss potential tradeoffs of each parametrization in different scenarios. Our main contributions are: 1) We first reveal that only four groups of stable closed-loop transfer matrices are equivalent to internal stability: one of them is used in the SLP, another one is used in the IOP, and the other two are new, leading to two new convex parameterizations of $C_{\rm stab}$. 2) We then investigate the properties of these parameterizations after imposing the finite impulse response (FIR) approximation, revealing that the IOP has the best ability of approximating $C_{\rm stab}$ given FIR constraints. 3) These four parameterizations require no a priori doubly-coprime factorization of the plant, but impose a set of equality constraints. However, these equality constraints will never be satisfied exactly in numerical computation. We prove that the IOP is numerically robust for open-loop stable plants, in the sense that small mismatches in the equality constraints do not compromise the closed-loop stability. The SLP is known to enjoy numerical robustness in the state feedback case; here, we show that numerical robustness of the four-block SLP controller requires case-by-case analysis in the general output feedback case.

Key words: Internal stability, Youla parameterization, System-level synthesis, Convex optimization.

1 Introduction

Feedback systems must be stable in some appropriate sense for practical deployment, and thus one fundamental problem in control theory is to design a feedback controller that stabilizes a given dynamical system [1]. Indeed, many control synthesis problems include stability as a constraint while optimizing some performance [2]. However, it is well-known that the set of stabilizing controllers is non-convex, and hence, hard to search directly over. One standard approach is to parameterize all stabilizing controllers and the corresponding closed-loop responses in a convex way, and then to optimize the performance over the new parameter(s) using convex optimization [3].

A classical method for parameterizing the set of all internally stabilizing controllers is based on the celebrated *Youla parameterization* [4] which relies on a doubly-coprime factorization of the system. It is shown in [3] that many

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performance specifications on the closed-loop system can be expressed in the Youla parameterization framework via convex optimization. Moreover, the foundational results of robust and optimal control are built on the Youla parameterization [1,5]. Recently, a system-level parameterization (SLP) [6] and an input-output parameterization (IOP) [7] were proposed to characterize the set of internally stabilizing controllers, without relying on the doubly-coprime factorization technique. In principle, Youla, the SLP and the IOP all directly treat certain closed-loop responses as design parameters, and thus shift the controller synthesis from the design of a controller to the design of the closed-loop responses. We note that an open-source Python-based implementation for the SLP and the IOP is available [8].

Besides the classical control synthesis problems [1, 5], closed-loop parameterizations are powerful tools in other areas, such as distributed optimal control [9–14] and quantifying the performance of learning in control [15–19]. In distributed control, the goal is to design sub-controllers that only rely on locally available information which is crucial for many cyber-physical systems. Enforcing these information constraints may make the control problems computationally intractable [20,21]. Yet, it is well-known that a notion of quadratic invariance (QI) [9,11,14] allows equivalently translating the information constraints on the controller to convex constraints on the Youla parameter, thus

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preserving the convexity of distributed controller synthesis. The QI notion can also be integrated with the SLP and the IOP, resulting in equivalent convex reformulations [6,7]. Together with a recent notion of *Sparsity Invariance* [22], these closed-loop parameterizations enable deriving convex approximations for problems with general sparsity constraints beyond QI; see [23, Remark 5] for example.

For learning in control, the SLP was recently integrated within a procedure called *Coarse-ID control* to derive a sample complexity bound for learning the classical linear quadratic regulator (LQR) [15]. This procedure was exploited in [16] to derive high probability guarantees of sublinear regret using an adaptive LQR control architecture. In [17, 18], based on the Youla parameterization, an online gradient descent algorithm was proposed to achieve sub-linear regret for learning the linear quadratic gaussian (LQG) controller. In [19], the Youla framework was used to derive a sample complexity bound on learning the *globally optimal* distributed controller subject to QI constraints. Note that the results in [17–19] clearly motivate the shift from static controllers to dynamic ones in complex machine-learning based control tasks.

While Youla [4], the SLP [6], and the IOP [7] are fundamental building blocks for distributed controller synthesis and learning-based control applications, a few critical issues have been left unexplored. First, it is shown that Youla, the SLP, and the IOP are all equivalent [23], but it remains unclear whether there exist other equivalent parameterizations using closed-loop responses beyond them. Second, the decision variables in these closed-loop parameterizations are in general infinite dimensional. In [6,7,24], the authors enforced finite impulse response (FIR) constraints on the decision variables for practical computation, leading to finite-dimensional convex optimization problems; however, the impact of the FIR approximation on different parameterizations has not been addressed before. Third, unlike Youla, the SLP and the IOP do not rely on computing a doubly-coprime factorization a priori, but instead introduce a set of equality constraints for achievable closed-loop responses. A fact that is not investigated in the SLP [6, 24] or the IOP [7] is that the set of equality constraints can never be satisfied exactly in numerical computation, potentially affecting the closed-loop stability.

1.1 Contributions

This paper aims to investigate the issues raised above and to provide a more complete understanding of closed-loop parameterizations: we introduce new parameterizations beyond SLP/IOP, and also discuss tradeoffs among these parameterizations in different scenarios. Specifically, the contributions of this paper are as follows.

First, we examine all possible parameterizations for the set of internally stabilizing controllers C_{stab} using closed-loop responses from the disturbances $(\delta_{\mathbf{x}}, \delta_{\mathbf{y}}, \delta_{\mathbf{u}})$ to state, output, control signals $(\mathbf{x}, \mathbf{y}, \mathbf{u})$; see Figure 1 for illustration. Our strategy is to examine the cases where the stability of external transfer matrices is equivalent to internal stability. We reveal that only four groups of stable disturbance-to-signal maps can guarantee internal stability (see Theorem 1): one of them is used in the SLP [6], another one is a classical result and is used in the IOP [7], and the other two

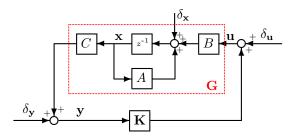


Figure 1. Interconnection of the plant G and the controller K.

have not been discussed before and thus can be used to derive two new parameterizations (Propositions 3 and 4). Our results are exclusive, in the sense that there are no other parameterizations for C_{stab} using closed-loop responses from $(\delta_{\mathbf{x}}, \delta_{\mathbf{y}}, \delta_{\mathbf{u}})$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$.

Second, we investigate the impact of imposing FIR constraints on the closed-loop parameterizations. We show that the IOP provides the tightest approximation of $\mathcal{C}_{\text{stab}}$ after imposing FIR constraints (Theorem 2). This result is enabled by the fact that the IOP directly deals with the maps from inputs to outputs without passing through the system state, while the SLP and the two new parametrizations explicitly involve the system state and/or the disturbance on the system state. Motivated by [25], we also characterize state-space realizations for the controllers in the closed-loop parameterizations after imposing the FIR approximation (Theorems 3 and 4). The state-space realizations provide easily implementable controllers for practical deployment.

Third, we prove that in the IOP framework, small numerical mismatches in the equality constraints do not compromise closed-loop stability for open-loop stable plants, but can destabilize the closed-loop system for unstable plants (Theorem 5). This result holds similarly for the two new closed-loop parameterizations. We also show that the fourblock SLP controller in the output-feedback case is potentially vulnerable to destabilization due to small mismatches in the equality constraints (Theorem 6), no matter whether the plant is open-loop stable or unstable. This issue is irrespective of the SLP controller implementation in [6, 24]. The classical Youla parameterization has no explicit equality constraints, and we show that a doubly-coprime factorization of the plant can be used to eliminate the equality constraints in the closed-loop parameterizations (Proposition 6).

1.2 Paper Structure

The rest of this paper is organized as follows. We state the problem in Section 2. The relationship between the stability of external transfer matrices and internal stability is revealed in Section 3. Four parameterizations of stabilizing controllers using closed-loop responses, including the SLP and the IOP, are presented in Section 4. Numerical computation using the FIR constraints and controller implementation are discussed in Section 5. We investigate the numerical robustness of closed-loop parameterizations in Section 6. A numerical application is shown in Section 7, and we conclude the paper in Section 8.

1.3 Notation

The symbols \mathbb{R} and \mathbb{N} refer to the set of real and integer numbers, respectively. We use lower and upper case letters (e.g. x and A) to denote vectors and matrices, respectively. Lower and upper case boldface letters (e.g. \mathbf{x} and \mathbf{G}) are used to denote signals and transfer matrices, respectively. We denote the set of real-rational proper stable ¹ transfer matrices as \mathcal{RH}_{∞} . We use the notation $\mathbf{G} \in \frac{1}{2}\mathcal{RH}_{\infty}$ to denote that **G** is stable and strictly proper. Given $\mathbf{G} \in \mathcal{RH}_{\infty}$, we denote its \mathcal{H}_{∞} norm by $\|\mathbf{G}\|_{\infty} := \sup_{\omega} \sigma_{\max}(\mathbf{G}(e^{j\omega})),$ where $\sigma_{\text{max}}(\cdot)$ denotes the maximum singular value. Given a stable transfer matrix $\mathbf{G}(z)$, the square of its \mathcal{H}_2 norm is $\|\mathbf{G}\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Trace} \left(\mathbf{G}^*(e^{j\omega})\mathbf{G}(e^{j\omega})\right) d\omega$. For simplicity, we omit the dimension of transfer matrices, which shall be clear in the context. Also, we use I (resp. 0) to denote the identity matrix (resp. zero matrix) of compatible dimension. In Section 5.2, to avoid ambiguity, we explicitly write the matrix dimension and use I_p to denote the identity matrix of dimension p. Finally, the state-space realiza-

tion
$$C(zI - A)^{-1}B + D$$
 is denoted as $\left| \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right|$.

2 Problem statement

2.1 System model

We consider strictly proper discrete-time linear timeinvariant (LTI) plants of the form

$$x[t+1] = Ax[t] + Bu[t] + \delta_x[t],$$

$$y[t] = Cx[t] + \delta_y[t],$$
(1)

where $x[t] \in \mathbb{R}^n, u[t] \in \mathbb{R}^m, y[t] \in \mathbb{R}^p$ are the state vector, control action, and measurement vector at time t, respectively; $\delta_x[t] \in \mathbb{R}^n$ and $\delta_y[t] \in \mathbb{R}^p$ are external disturbances on the state and measurement vectors at time t, respectively. The transfer matrix from \mathbf{u} to \mathbf{y} is $\mathbf{G} = C(zI - A)^{-1}B$. Consider an LTI dynamical controller

$$\mathbf{u} = \mathbf{K}\mathbf{y} + \boldsymbol{\delta}_u, \tag{2}$$

where δ_u is the external disturbance on the control action. A state-space realization of (2) is

$$\xi[t+1] = A_k \xi[t] + B_k y[t], u[t] = C_k \xi[t] + D_k y[t] + \delta_u[t],$$
(3)

where $\xi[t] \in \mathbb{R}^q$ is the internal state of the controller at time t. The formulation (3) above reduces to a static controller when $(A_k, B_k, C_k, D_k) = (0, 0, 0, K)$ for some $K \in \mathbb{R}^{m \times p}$. In this paper, we make the following standard assumption. **Assumption 1** Both the plant and controller realizations are stabilizable and detectable, i.e., (A, B) and (A_k, B_k) are stabilizable, and (A, C) and (A_k, C_k) are detectable.

Applying the controller (2) to the plant (1) leads to a closed-loop system shown in Fig. 1. Since the plant is strictly proper, the closed-loop system is always well-posed [1, Lemma 5.1].

2.2 Internal stability

Internal stability is defined as follows [1, Chapter 5.3]:

Definition 1 The system in Fig. 1 is internally stable if it is well-posed, and the states $(x[t], \xi[t])$ converge to zero as $t \to \infty$ for all initial states $(x[0], \xi[0])$ when $\delta_x[t] = 0, \delta_y[t] = 0, \delta_y[t] = 0, \forall t$.

We say the controller K internally stabilizes the plant G if the closed-loop system in Fig. 1 is internally stable. The set of all LTI internally stabilizing controllers is defined as

$$C_{\text{stab}} := \{ \mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{G} \}.$$
 (4)

Note that when an infinite time-horizon is considered, a feedback system must at least be stable, and any controller synthesis will implicitly or explicitly involve a constraint $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$. Therefore, it is fundamentally important to characterize $\mathcal{C}_{\mathrm{stab}}$. Indeed, it is well-known that $\mathcal{C}_{\mathrm{stab}}$ is nonconvex, and it is not difficult to find explicit examples where $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{C}_{\mathrm{stab}}$ and $\frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2) \notin \mathcal{C}_{\mathrm{stab}}$. Accordingly, it is not easy to directly search over $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$ for control synthesis, and a suitable change of variables is used in many control synthesis procedures [1–7, 23].

A standard state-space characterization of internal stabilization is as follows.

Lemma 1 ([1, Lemma 5.2]) Under Assumption 1, K internally stabilizes G if and only if

$$A_{\rm cl} := \begin{bmatrix} A + BD_k C & BC_k \\ B_k C & A_k \end{bmatrix} \tag{5}$$

is stable.

Note that the result in Lemma 1 is a simplified version of [1, Lemma 5.2] because we focus on strictly proper plants for simplicity. Lemma 1 leads to an explicit state-space characterization of the set $\mathcal{C}_{\mathrm{stab}}$ as follows:

$$C_{\text{stab}} = \left\{ \mathbf{K} \mid A_{\text{cl}} = \begin{bmatrix} A + BD_k C & BC_k \\ B_k C & A_k \end{bmatrix} \text{ is stable} \right\}, \quad (6)$$

where $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. Unfortunately, the stability condition on A_{cl} in (6) is still non-convex in terms of the parameters A_k, B_k, C_k, D_k .

2.3 Frequency-domain characterizations

Unlike the state-space parameterization (6), there are several frequency-domain characterizations for C_{stab} , where only convex constraints are involved in the new parameters. A classical approach is the celebrated *Youla parameterization* [4], where a doubly-coprime factorization of the plant is used.

Definition 2 A collection of stable transfer matrices, $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r \in \mathcal{RH}_{\infty}$ is called a doubly-coprime factorization of \mathbf{G} if $\mathbf{P}_{22} = \mathbf{N}_r \mathbf{M}_r^{-1} = \mathbf{M}_l^{-1} \mathbf{N}_l$ and

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = I.$$

Such doubly-coprime factorization can always be computed if the state-space realization of ${\bf G}$ is stabilizable and de-

 $^{^1\,}$ Throughout the paper, "stable" means "asymptotically stable", i.e., all eigenvalues/poles have strictly negative real parts.

tectable [26]. We have the following equivalence [4]

$$C_{\text{stab}} = \{ \mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) (\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_{\infty} \},$$
(7)

where \mathbf{Q} is denoted as the Youla parameter. Note that the Youla parameter \mathbf{Q} can be freely chosen in \mathcal{RH}_{∞} . We refer the interested reader to [1,4,5] for more details on the Youla parameterization.

Two recent approaches are the system-level parameterization (SLP) [6] and the input-output parameterization (IOP) [7], where no doubly-coprime factorization is required a priori. Both the SLP and the IOP use certain closed-loop responses for parameterizing C_{stab} . Inspired by the SLP and the IOP, we aim to investigate all possible parameterizations for C_{stab} using the closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ (Sections 3 and 4), which we denote as closed-loop parameterizations in this paper. In addition, we further investigate the computational properties of closed-loop parameterizations, including numerical computation and implementation (Section 5), as well as numerical robustness (Section 6).

We conclude this section by stating the following classical result, which will be frequently used.

Lemma 2 ([1, Chapter 3]) Given a transfer matrix $T(z) = C(zI - A)^{-1}B + D$, we have

- If (A, B, C) is detectable and stabilizable, then $\mathbf{T}(z) \in \mathcal{RH}_{\infty}$ if and only if A is stable;
- If (A, B) is not stabilizable, or (A, C) is not detectable, then the stability of A is sufficient but not necessary for $\mathbf{T}(z) \in \mathcal{RH}_{\infty}$.

3 External transfer matrix characterization of internal stability

In this section, we revisit the external transfer matrix characterization of internal stability, which will be applied to characterize $\mathcal{C}_{\mathrm{stab}}$ in the next section.

Combining (1) with (2), we can write the closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_{x} \\ \boldsymbol{\delta}_{y} \\ \boldsymbol{\delta}_{u} \end{bmatrix}, \tag{8}$$

where we have $\Phi_{xx} = (zI - A - BKC)^{-1}$ and

$$\Phi_{xy} = \Phi_{xx}BK, \qquad \Phi_{xu} = \Phi_{xx}B,
\Phi_{yx} = C\Phi_{xx}, \qquad \Phi_{yy} = C\Phi_{xx}BK + I,
\Phi_{yu} = C\Phi_{xx}B, \qquad \Phi_{ux} = KC\Phi_{xx},
\Phi_{uy} = K(C\Phi_{xx}BK + I), \qquad \Phi_{uu} = KC\Phi_{xx}B + I.$$
(9)

We define the closed-loop response transfer matrix as

$$\mathbf{\Phi} := \begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix}. \tag{10}$$

A notion of external transfer matrix stability is defined as follows.

Definition 3 ([1, Chapter 5]) The closed-loop system is disturbance-to-signal stable if the closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ are all stable, i.e., $\mathbf{\Phi} \in \mathcal{RH}_{\infty}$.

$3.1 \quad General \ plant$

Under Assumption 1, it is known that the internal stability in Definition 1 and the disturbance-to-signal stability in Definition 2 are equivalent [1, Chapter 5], *i.e.*, we have

$$C_{\text{stab}} = \{ \mathbf{K} \mid \mathbf{\Phi} \in \mathcal{RH}_{\infty}, \text{ with } \mathbf{\Phi} \text{ defined in (10)} \}.$$
 (11)

In fact, it is sufficient to enforce a subset of elements in Φ to be stable, as shown in [1, Lemma 5.3].

Lemma 3 Under Assumption 1, **K** internally stabilizes **G** if and only if the closed-loop responses from $(\boldsymbol{\delta}_y, \boldsymbol{\delta}_u)$ to (\mathbf{y}, \mathbf{u}) are stable, i.e.,

$$egin{bmatrix} oldsymbol{\Phi}_{yy} & oldsymbol{\Phi}_{yu} \ oldsymbol{\Phi}_{uy} & oldsymbol{\Phi}_{uu} \end{bmatrix} \in \mathcal{RH}_{\infty}.$$

For notational simplicity, we denote

$$\left(\begin{bmatrix} oldsymbol{\delta}_y \\ oldsymbol{\delta}_u \end{bmatrix}
ightarrow \begin{bmatrix} oldsymbol{y} \\ oldsymbol{u} \end{bmatrix} := \begin{bmatrix} oldsymbol{\Phi}_{yy} & oldsymbol{\Phi}_{yu} \\ oldsymbol{\Phi}_{uy} & oldsymbol{\Phi}_{uu} \end{bmatrix}.$$

The result in Lemma 3 motivates the question of whether we can select different minimal sets of elements in Φ for internal stability. For example, if the closed-loop responses from (δ_x, δ_y) to (\mathbf{x}, \mathbf{y}) are stable, *i.e.*,

$$\left(\begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right) := \begin{bmatrix} \boldsymbol{\Phi}_{xx} & \boldsymbol{\Phi}_{xy} \\ \boldsymbol{\Phi}_{yx} & \boldsymbol{\Phi}_{yy} \end{bmatrix} \in \mathcal{RH}_{\infty},$$

can we guarantee that the closed-loop system is internally stable? The answer is negative, as proved in Theorem 1 below.

In particular, we consider all possible combinations of four closed-loop responses that may guarantee internal stability. When choosing two disturbances and two outputs from (8), we have in total $\binom{3}{2} \times \binom{3}{2} = 9$ choices, *i.e.*,

$$\begin{pmatrix}
\begin{bmatrix} \delta_{x} \\ \delta_{y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{pmatrix} \begin{bmatrix} \delta_{x} \\ \delta_{y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}, \begin{pmatrix} \begin{bmatrix} \delta_{x} \\ \delta_{y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \\
\begin{pmatrix} \begin{bmatrix} \delta_{y} \\ \delta_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{pmatrix} \begin{bmatrix} \delta_{y} \\ \delta_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}, \begin{pmatrix} \begin{bmatrix} \delta_{y} \\ \delta_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}, \\
\begin{pmatrix} \begin{bmatrix} \delta_{x} \\ \delta_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{pmatrix} \begin{bmatrix} \delta_{x} \\ \delta_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}, \begin{pmatrix} \begin{bmatrix} \delta_{x} \\ \delta_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}.
\end{pmatrix}$$
(12)

Note that it is in general not sufficient to select less than four close-loop responses since there are two dynamical parts in system (1) and controller (3). One main result of this section shows that the stability of any of the groups of four closed-loop responses in the top-right corner of (12), highlighted in blue, is equivalent to internal stability.

Theorem 1 Consider the LTI system (1), evolving under a dynamic control policy (3). Under Assumption 1, the fol-

lowing statements are equivalent:

(1) **K** internally stabilizes **G**;

$$(2) \ \left(\begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_{\infty};$$

$$(3) \left(\begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \to \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_{\infty};$$

$$(4) \left(\begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_{\infty};$$

$$(5) \ \left(\begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_{\infty}.$$

Moreover, the stability of any other group of four closed-loop responses in (12) is not sufficient for internal stability.

Proof: The idea of our proof is to use a state-space representation of the closed-loop system, which is motivated by [1, Lemma 5.3]. From (1) and (3), we have

$$\begin{bmatrix} x[t+1] \\ \xi[t+1] \end{bmatrix} = \begin{bmatrix} A \\ A_k \end{bmatrix} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} B \\ B_k \end{bmatrix} \begin{bmatrix} u[t] \\ y[t] \end{bmatrix} + \begin{bmatrix} \delta_x[t] \\ 0 \end{bmatrix}, (13)$$

and

$$\begin{bmatrix} I - D_k \\ 0 & I \end{bmatrix} \begin{bmatrix} u[t] \\ y[t] \end{bmatrix} = \begin{bmatrix} 0 & C_k \\ C & 0 \end{bmatrix} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} \delta_u[t] \\ \delta_y[t] \end{bmatrix}. \tag{14}$$

Substituting (14) into (13) leads to

$$\begin{bmatrix} x[t+1] \\ \xi[t+1] \end{bmatrix} = A_{\rm cl} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} B & BD_k \\ 0 & B_k \end{bmatrix} \begin{bmatrix} \delta_u[t] \\ \delta_y[t] \end{bmatrix} + \begin{bmatrix} \delta_x[t] \\ 0 \end{bmatrix},$$

i.e., we have

$$\begin{bmatrix}
x[t+1] \\
\xi[t+1]
\end{bmatrix} = A_{cl} \begin{bmatrix} x[t] \\
\xi[t] \end{bmatrix} + \begin{bmatrix} I & BD_k & B \\
0 & B_k & 0 \end{bmatrix} \begin{bmatrix} \delta_x[t] \\
\delta_y[t] \\
\delta_u[t] \end{bmatrix},
\begin{bmatrix}
x[t] \\
y[t] \\
u[t] \end{bmatrix} = \begin{bmatrix} I & 0 \\
C & 0 \\
D_k C & C_k \end{bmatrix} \begin{bmatrix} x[t] \\
\xi[t] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\
0 & I & 0 \\
0 & D_k & I \end{bmatrix} \begin{bmatrix} \delta_x[t] \\
\delta_y[t] \\
\delta_u[t] \end{bmatrix}.$$
(15)

Therefore, the closed-loop responses from $(\boldsymbol{\delta}_x, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u) \rightarrow (\mathbf{x}, \mathbf{y}, \mathbf{u})$ are

$$\begin{bmatrix} I & 0 \\ C & 0 \\ D_k C & C_k \end{bmatrix} (zI - A_{cl})^{-1} \begin{bmatrix} I & BD_k & B \\ 0 & B_k & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & D_k & I \end{bmatrix}, (16)$$

from which, we get state-space realizations of the following

closed-loop responses

$$\begin{pmatrix}
\begin{bmatrix} \boldsymbol{\delta}_{x} \\ \boldsymbol{\delta}_{y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \end{pmatrix} = \hat{C}_{1}(zI - A_{\text{cl}})^{-1}\hat{B}_{1} + \begin{bmatrix} 0 & I \\ 0 & D_{k} \end{bmatrix}, \quad (17a)$$

$$\begin{pmatrix}
\begin{bmatrix} \boldsymbol{\delta}_{x} \\ \boldsymbol{\delta}_{y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \end{pmatrix} = \hat{C}_{2}(zI - A_{\text{cl}})^{-1}\hat{B}_{1} + \begin{bmatrix} 0 & 0 \\ 0 & D_{k} \end{bmatrix}, \quad (17b)$$

$$\begin{pmatrix}
\begin{bmatrix} \boldsymbol{\delta}_{y} \\ \boldsymbol{\delta}_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \end{pmatrix} = \hat{C}_{1}(zI - A_{\text{cl}})^{-1}\hat{B}_{2} + \begin{bmatrix} I & 0 \\ D_{k} & I \end{bmatrix}, \quad (17c)$$

$$\begin{pmatrix}
\begin{bmatrix} \boldsymbol{\delta}_{y} \\ \boldsymbol{\delta}_{u} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \end{pmatrix} = \hat{C}_{2}(zI - A_{\text{cl}})^{-1}\hat{B}_{2} + \begin{bmatrix} 0 & 0 \\ D_{k} & I \end{bmatrix}, \quad (17d)$$

where

$$\hat{B}_{1} = \begin{bmatrix} I & BD_{k} \\ 0 & B_{k} \end{bmatrix}, \quad \hat{B}_{2} = \begin{bmatrix} BD_{k} & B \\ B_{k} & 0 \end{bmatrix},
\hat{C}_{1} = \begin{bmatrix} C & 0 \\ D_{k}C & C_{k} \end{bmatrix}, \quad \hat{C}_{2} = \begin{bmatrix} I & 0 \\ D_{k}C & C_{k} \end{bmatrix}.$$
(18)

By Lemma 1, we know that **K** internally stabilizes **G** if and only if the closed-loop matrix $A_{\rm cl}$ defined in (5) is stable. It is obvious true that $1) \Rightarrow 2$, $1) \Rightarrow 3$, $1) \Rightarrow 4$, and $1) \Rightarrow 5$.

Next, we prove if anyone of 2) - 5) is true, the matrix $A_{\rm cl}$ is stable. According to Lemma 2, it remains to prove that the state-space realizations (17a)–(17d) are all stabilizable and detectable. This is equivalent to showing that $(A_{\rm cl}, \hat{B}_1), (A_{\rm cl}, \hat{B}_2)$ are stabilizable and that $(A_{\rm cl}, \hat{C}_1), (A_{\rm cl}, \hat{C}_2)$ are detectable. We let

$$\hat{F}_1 = \begin{bmatrix} F & 0 \\ -C & F_k \end{bmatrix}$$

where F and F_k are chosen such that A+F and $A_k+B_kF_k$ are stable. Then, we have that

$$A_{\rm cl} + \hat{B}_1 \hat{F}_1 = \begin{bmatrix} A + F & BC_k + BD_k F_k \\ 0 & A_k + B_k F_k \end{bmatrix}$$

is stable, and thus $(A_{\rm cl}, \hat{B}_1)$ is stablizable. Similar arguments show that $(A_{\rm cl}, \hat{B}_2)$ is stabilizable, and $(A_{\rm cl}, \hat{C}_1)$, $(A_{\rm cl}, \hat{C}_2)$ are detectable.

For the second part of Theorem 1, we first prove that the stability of $\begin{pmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \end{pmatrix}$ is not sufficient for internal stability. From (16), a state-space realization of the closed-

stability. From (16), a state-space realization of the closed-loop responses from $(\boldsymbol{\delta}_x, \boldsymbol{\delta}_y)$ to (\mathbf{x}, \mathbf{y}) is

$$\left(\begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \to \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right) = \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix} (zI - A_{\mathrm{cl}})^{-1} \hat{B}_1 + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Since
$$\begin{pmatrix} A_{\rm cl}, \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix} \end{pmatrix}$$
 is not detectable in general, the stabil-

ity of
$$\begin{pmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \end{pmatrix}$$
 cannot guarantee the stability of A_{cl} .

Therefore, it is not sufficient for internal stability either. The other claims can be proved in a similar way: the corresponding state-space realization of the closed-loop transfer matrix is not stabilizable and/or detectable.

As shown in Theorem 1, to guarantee internal stability for general plants, it is always required to select δ_y as an input and \mathbf{u} as an output, leading to four possible groups of closed-loop responses. The groups of closed-loop responses in (12), except those in blue, do not have either δ_y or \mathbf{u} , and thus fail to guarantee internal stability. Note that Theorem 1 is exclusive in the sense that there exist no other combinations of stable closed-loop responses that are equivalent to internal stability, and Lemma 3 is included as the equivalence between 1) and 4) in Theorem 1.

3.2 Two special cases: Stable plants and State feedback

Here, we show that the transfer matrix characterization of internal stability can be simplified for special cases: 1) open-loop stable plants; 2) the state feedback case. To guarantee internal stability, instead of considering four closed-loop responses in Theorem 1, the stability of one particular closed-loop response is sufficient in the case of open-loop stable plants, and the stability of two particular closed-loop responses is sufficient in the state feedback case.

The following result is classical, which is the same as [1, Corollary 5.5]. For completeness, we provide a proof from a state-space perspective.

Corollary 1 Consider the LTI system (1), evolving under a dynamic control policy (3). If the LTI system is openloop stable (i.e., A is stable), then $\mathbf{K} \in \mathcal{C}_{stab}$ if and only if $(\boldsymbol{\delta}_y \to \mathbf{u}) := \boldsymbol{\Phi}_{uy} \in \mathcal{RH}_{\infty}$.

Proof: The "only if" direction is true by definition. We now prove the sufficiency. From (16), we have

$$\mathbf{\Phi}_{uy} = \left[D_k C \ C_k \right] (zI - A_{\mathrm{cl}})^{-1} \begin{bmatrix} BD_k \\ B_k \end{bmatrix} + D_k.$$

Considering the fact that the following matrix

$$A_{\rm cl} + \begin{bmatrix} BD_k \\ B_k \end{bmatrix} \begin{bmatrix} -C \ F_k \end{bmatrix} = \begin{bmatrix} A \ BC_k + BD_k F_k \\ 0 \ A_k + B_k F_k \end{bmatrix},$$

is stable when A and $A_k + B_k F_k$ are stable, we know that $\begin{pmatrix} A_{\rm cl}, \begin{bmatrix} BD_k \\ B_k \end{bmatrix} \end{pmatrix}$ is stabilizable. Similarly, we can show that $\begin{pmatrix} A_{\rm cl}, \begin{bmatrix} D_k C_k C_k \end{bmatrix} \end{pmatrix}$ is detectable. Therefore, if $\Phi_{\rm cl} \in \mathcal{RH}$

 $\left(A_{\text{cl}}, \left\lfloor D_k C \ C_k \right\rfloor \right)$ is detectable. Therefore, if $\Phi_{uy} \in \mathcal{RH}_{\infty}$, we have A_{cl} is stable, meaning that $\mathbf{K} \in \mathcal{C}_{\text{stab}}$.

In the state feedback case, we have the following result. Corollary 2 Consider the LTI system (1), evolving under a dynamic control policy (3). If C = I, then $\mathbf{K} \in \mathcal{C}_{stab}$ if

and only if
$$\left(oldsymbol{\delta}_x
ightarrow egin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) := egin{bmatrix} oldsymbol{\Phi}_{xx} \\ oldsymbol{\Phi}_{ux} \end{bmatrix} \in \mathcal{RH}_{\infty}.$$

Proof: When C = I, from (16), we have

$$\begin{bmatrix} \mathbf{\Phi}_{xx} \\ \mathbf{\Phi}_{ux} \end{bmatrix} = \begin{bmatrix} I & 0 \\ D_k & C_k \end{bmatrix} \left(zI - \begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix} \right)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

From the proof of Theorem 1, we know

$$\left(\begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix}, \begin{bmatrix} I & 0 \\ D_k & C_k \end{bmatrix} \right)$$

is detectable. In Appendix A, we prove that

$$\left(\begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix}\right)$$
(19)

is stabilizable. Therefore, if $\begin{bmatrix} \mathbf{\Phi}_{xx} \\ \mathbf{\Phi}_{ux} \end{bmatrix} \in \mathcal{RH}_{\infty}$, we have A_{cl} is stable, meaning that $\mathbf{K} \in \mathcal{C}_{\text{stab}}$.

The result in Corollary 2 has been used in the state feedback case of the system-level parametrization [6]. The proof in [6] used a frequency-based method. Here, we provided an alternative proof from a state-space perspective, which is consistent with the proofs for Theorem 1 and Corollary 1.

4 Parameterizations of stabilizing controllers

The results in Theorem 1 can be used to parameterize the set of internally stabilizing controllers $\mathcal{C}_{\mathrm{stab}}$, leading to four equivalent parameterizations. One of them corresponds to the SLP [6], and another one is the IOP [7]. The remaining two parameterizations are new and, to the best of the authors' knowledge, have not been characterized before. The results in Corollaries 1 and 2 can also be used to parameterize $\mathcal{C}_{\mathrm{stab}}$ in a simplified way.

4.1 Four equivalent parameterizations for general plants

The closed-loop responses from (δ_x, δ_y) to (\mathbf{x}, \mathbf{u}) have been utilized in the SLP [6]. Specifically, consider

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix}. \tag{20}$$

We have the following system-level parameterization (SLP). **Proposition 1 (SLP [6, Theorem 2])** Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:

(1) For any $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$, the resulting closed-loop responses (20) are in the following affine subspace

$$\begin{bmatrix} zI - A - B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$\mathbf{\Phi}_{xx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy} \in \mathcal{RH}_{\infty}.$$
(21)

(2) For any transfer matrices Φ_{xx} , Φ_{ux} , Φ_{xy} , Φ_{uy} satisfying (21), $\mathbf{K} = \Phi_{uy} - \Phi_{ux}\Phi_{xx}^{-1}\Phi_{xy} \in \mathcal{C}_{stab}$.

We refer to $\mathbf{K} = \mathbf{\Phi}_{uy} - \mathbf{\Phi}_{ux}\mathbf{\Phi}_{xx}^{-1}\mathbf{\Phi}_{xy}$ as the four-block SLP controller. Also, the closed-loop responses from $(\boldsymbol{\delta}_y, \boldsymbol{\delta}_u)$ to (\mathbf{y}, \mathbf{u}) have been used in the IOP [7]. Specifically, consider

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix} . \tag{22}$$

We have the following input-output parameterization (IOP). **Proposition 2 (IOP [7, Theorem 1])** Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:

(1) For any $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$, the resulting closed-loop responses (22) are in the following affine subspace

$$\begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$\mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uu} \in \mathcal{RH}_{\infty}.$$

$$(23)$$

(2) For any transfer matrices Φ_{yy} , Φ_{uy} , Φ_{yu} , Φ_{uu} satisfying (23), $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1} \in \mathcal{C}_{stab}$.

Next, we consider the following closed-loop responses

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix}. \tag{24}$$

We have the following result about a new parametrization of $\mathcal{C}_{\mathrm{stab}}$.

Proposition 3 (Mixed I) Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:

(1) For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses (24) are in the following affine subspace

$$\begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} = \begin{bmatrix} C(zI - A)^{-1} & I \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = 0,$$

$$\mathbf{\Phi}_{yx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy} \in \mathcal{RH}_{\infty}.$$
(25)

(2) For any transfer matrices Φ_{yx} , Φ_{ux} , Φ_{yy} , Φ_{uy} satisfying (25), $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1} \in \mathcal{C}_{stab}$.

Proof: Statement 1: Given any $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$, it is easy to derive that the closed-loop responses (24) are

$$\Phi_{yx} = (I - \mathbf{G}\mathbf{K})^{-1}C(zI - A)^{-1},$$

$$\Phi_{yy} = (I - \mathbf{G}\mathbf{K})^{-1},$$

$$\Phi_{ux} = \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}C(zI - A)^{-1},$$

$$\Phi_{uy} = \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1},$$

which are all stable by definition. Then, it is not difficult to verify that

$$\Phi_{yx} - \mathbf{G}\Phi_{ux} = C(zI - A)^{-1},$$

$$\Phi_{yy} - \mathbf{G}\Phi_{uy} = (I - \mathbf{G}\mathbf{K})^{-1} - \mathbf{G}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} = I,$$

and that

$$\mathbf{\Phi}_{yx}(zI - A) - \mathbf{\Phi}_{yy}C = 0,$$

$$\mathbf{\Phi}_{ux}(zI - A) - \mathbf{\Phi}_{uy}C = 0.$$

Therefore, the closed-loop responses Φ_{yx} , Φ_{yy} , Φ_{ux} , Φ_{uy} satisfy (25).

Statement 2: Consider any Φ_{yx} , Φ_{yy} , Φ_{ux} , Φ_{uy} satisfying (25). Since $\Phi_{yy} = I + \mathbf{G}\Phi_{uy}$ and \mathbf{G} is strictly proper, we know that Φ_{yy} is always invertible. Let $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1}$. We now verify the resulting closed-loop responses in (24) are all stable. In particular, we have

$$\mathbf{y} = (I - \mathbf{G}\mathbf{K})^{-1}C(zI - A)^{-1}\boldsymbol{\delta}_x,$$

and with $\mathbf{K} = \mathbf{\Phi}_{uy} \mathbf{\Phi}_{yy}^{-1}$, we have

$$(I - \mathbf{G}\mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1})^{-1}C(zI - A)^{-1}$$

$$= \mathbf{\Phi}_{yy}(\mathbf{\Phi}_{yy} - \mathbf{G}\mathbf{\Phi}_{uy})^{-1}C(zI - A)^{-1}$$

$$= \mathbf{\Phi}_{yy}C(zI - A)^{-1}$$

$$= \mathbf{\Phi}_{yx} \in \mathcal{RH}_{\infty},$$

where the equalities follow from the fact that Φ_{yx} , Φ_{yy} , Φ_{ux} , Φ_{uy} satisfy (25). Also, we have that

$$\mathbf{y} = (I - \mathbf{G}\mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1})^{-1}\boldsymbol{\delta}_y = \mathbf{\Phi}_{yy}\boldsymbol{\delta}_y.$$

Similarly, we can show that

$$\mathbf{u} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}C_2(zI - A)^{-1}\boldsymbol{\delta}_x = \boldsymbol{\Phi}_{ux}\boldsymbol{\delta}_x,$$

$$\mathbf{u} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\boldsymbol{\delta}_u = \boldsymbol{\Phi}_{uy}\boldsymbol{\delta}_u.$$

Therefore, we have proved that $\begin{pmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \end{pmatrix} \in \mathcal{RH}_{\infty}$, using the controller $\mathbf{K} = \boldsymbol{\Phi}_{uy} \boldsymbol{\Phi}_{yy}^{-1}$. By Theorem 1, we know $\mathbf{K} = \boldsymbol{\Phi}_{uy} \boldsymbol{\Phi}_{yy}^{-1} \in \mathcal{C}_{\mathrm{stab}}$.

Finally, we consider the case

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix}. \tag{26}$$

The following result mirrors Proposition 3 for an additional new parametrization of C_{stab} .

Proposition 4 (Mixed II) Consider the LTI system (1), evolving under a dynamic control policy (3). The following

statements are true:

(1) For any $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$, the resulting closed-loop responses (26) are in the following affine subspace

$$\begin{bmatrix} zI - A - B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} = 0,$$

$$\begin{bmatrix} \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix},$$

$$\mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uu} \in \mathcal{RH}_{\infty}.$$

$$(27)$$

(2) For any transfer matrices Φ_{xy} , Φ_{uy} , Φ_{xu} , Φ_{uu} satisfying (27), $\mathbf{K} = \Phi_{uu}^{-1} \Phi_{uy} \in \mathcal{C}_{\text{stab}}$.

The proof of Proposition 4 is similar to that of Proposition 3^2 , which is provided in Appendix B.

To summarize, Propositions 1–4 establish four equivalent methods to parameterize the set of internally stabilizing controllers using closed-loop responses:

$$\mathcal{C}_{\text{stab}} = \{ \mathbf{K} = \mathbf{\Phi}_{uy} - \mathbf{\Phi}_{ux} \mathbf{\Phi}_{xx}^{-1} \mathbf{\Phi}_{xy} \mid \mathbf{\Phi}_{xx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy} \\ \text{are in the affine subspace (21)} \}, \\
\mathcal{C}_{\text{stab}} = \{ \mathbf{K} = \mathbf{\Phi}_{uy} \mathbf{\Phi}_{yy}^{-1} \mid \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uu} \\ \text{are in the affine subspace (23)} \}, \\
\mathcal{C}_{\text{stab}} = \{ \mathbf{K} = \mathbf{\Phi}_{uy} \mathbf{\Phi}_{yy}^{-1} \mid \mathbf{\Phi}_{yx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy} \\ \text{are in the affine subspace (25)} \}, \\
\mathcal{C}_{\text{stab}} = \{ \mathbf{K} = \mathbf{\Phi}_{uu}^{-1} \mathbf{\Phi}_{uy} \mid \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xu}, \mathbf{\Phi}_{uu} \\ \text{are in the affine subspace (27)} \}.$$

Unlike the state-space characterization (6), the constraints (21), (23), (25), and (27) are all affine in the new parameters. Based on (21), (23), (25), and (27), convex optimization problems can be derived for the classical optimal controller synthesis; see [6,7,23] for details. We will present a case study in Section 7.

Remark 1 (Equivalence with Youla) The explicit equivalence between Propositions 1 & 2 and the Youla parameterization has been derived in [23]. It is not difficult to derive the explicit relationship between Propositions 3 & 4 and the Youla parameterization (7) using the approach of [7,23]. Besides, while there are four parameters in (21), (23), (25), or (27), there is only one freedom due to the affine constraints. This is consistent with the Youla parameterization, where only one parameter is involved with no explicit affine constraints. In Proposition 6, we will show that any doubly-coprime factorization of the plant can exactly eliminate the affine constraints (21), (23), (25), and (27).

Remark 2 (Numerical computation) Note that while being convex, the decision variables in (21), (23), (25), and (27) are infinite-dimensional. Thus, finite-dimensional approximations are needed for numerical computations, which will be discussed in details in Section 5. Finally, the affine constraints (21), (23), (25), and (27) can never be exactly satisfied in numerical computation, and Section 6

will formally discuss the issue of numerical robustness.

4.2 Two special cases: stable plants and state feedback

The results in Corollaries 1 and 2 can be exploited to derive simplified versions of Propositions 1–4. We will later show that these simplified parametrizations enjoy provable numerical robustness. When the plant is open-loop stable, the IOP (Proposition 2) and the Mixed I (Proposition 3) are simplified as follows.

Corollary 3 Consider the LTI system (1), evolving under a dynamic controller policy (3). If the LTI system is openloop stable, then we have

$$\mathcal{C}_{\mathrm{stab}} = \left\{ \mathbf{K} = \mathbf{\Phi}_{uy} \mathbf{\Phi}_{yy}^{-1} \left| \begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{uy} \end{bmatrix} = I, \ \mathbf{\Phi}_{uy} \in \mathcal{RH}_{\infty} \right\}.$$

This result is consistent with the classical one in [1, Theorem 12.7]. Note that for open-loop stable plants, the transfer matrix Φ_{uy} from the measurement disturbance δ_y to the control input \mathbf{u} is the same as the Youla parameter. Under the condition in Corollary 3, the Mixed II (Proposition 4) can be simplified as well:

$$\mathcal{C}_{ ext{stab}} \!=\! \left\{ \mathbf{K} = \mathbf{\Phi}_{uu}^{-1} \mathbf{\Phi}_{uy} \left| \begin{bmatrix} \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = I, \mathbf{\Phi}_{uy} \in \mathcal{RH}_{\infty} \right\}.$$

If the state is directly measurable for control, i.e., C = I, Corollary 2 leads to the following simplified version of SLP. Corollary 4 ([6, Theorem 1]) Consider the LTI system (1), evolving under a dynamic controller policy (3). If C = I, then we have

$$C_{\text{stab}} = \left\{ \mathbf{K} = \mathbf{\Phi}_{ux} \mathbf{\Phi}_{xx}^{-1} \middle| \begin{bmatrix} zI - A - B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{xx} \\ \mathbf{\Phi}_{ux} \end{bmatrix} = I, \\ \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{xx} \in \frac{1}{z} \mathcal{R} \mathcal{H}_{\infty} \right\}.$$

Note that the simplified IOP/Mixed I/Mixed II requires the stability of only one parameter, while the simplified SLP requires the stability of two parameters. The proofs for Corollaries 3 and 4 are similar to that of Proposition 3.

5 Numerical computation and controller implementation

This section investigates the numerical computation and controller implementation using the closed-loop parameterization for $\mathcal{C}_{\mathrm{stab}}$. As noted in Remark 2, since the decision variables in the affine constraints (21), (23), (25), and (27) are infinite-dimensional, it is not immediately obvious to derive efficient numerical computation to search over the feasible region. One practical method is to apply the finite impulse response (FIR) approximation, which is extensively used in [6, 7]. As we will see, the SLP, the IOP and the two new mixed parametrizations are not equivalent to each other after imposing FIR constraints. In this section, we also present standard state-space realizations (3) for the controllers using closed-loop responses.

² Note that both the SLP in Proposition 1 and the IOP in Proposition 2 can be proved similarly. The interested reader may easily verify the proof.

5.1 Numerical computation via FIR

We denote the space of finite impulse response (FIR) transfer matrices with horizon T as

$$\mathcal{F}_T := \left\{ \mathbf{H} \in \mathcal{RH}_{\infty} \,\middle|\, \mathbf{H} = \sum_{k=0}^T \frac{1}{z^k} H_k \right\},$$

where H_k denotes the *i*-th spectral component of the FIR transfer matrix \mathbf{H} . It is known that on letting the FIR length T go to infinity, \mathcal{F}_T becomes the same as \mathcal{RH}_{∞} [27, Theorem 4.7]. It is not difficult to check that after imposing the decision variables to be FIR transfer matrices of horizon T, the constraints (21), (23), (25), and (27) all become finite-dimensional affine constraints in terms of the spectral components of the closed-loop responses. Thus, searching for an internally stabilizing controller only requires to solve a linear program (LP) under the FIR assumption 3 .

Here, we show that imposing the FIR assumption has different effects depending on the chosen closed-loop parametrization.

Theorem 2 Given the LTI system (1), evolving under a dynamic control policy (3), we consider the statements:

(i)
$$\Phi \in \mathcal{F}_T$$
;

$$(ii) \left(\begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \to \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T; \tag{SLP}$$

(iii)
$$\left(\begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}\right) \in \mathcal{F}_T;$$
 (Mixed I)

$$(iv) \left(\begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix} \to \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T; \qquad (Mixed II)$$

$$(v) \left(\begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T. \tag{IOP}$$

If (A, B, C) and (A_k, B_k, C_k) are both stabilizable and detectable, we have $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (v)$ and $(i) \Leftrightarrow (ii) \Rightarrow (iv) \Rightarrow (v)$. In addition, if (A, B, C) and (A_k, B_k, C_k) are both controllable and observable, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.

Proof: The direction $(i) \Rightarrow (ii), (iii), (iv), (v)$ is true by definition. For any controller **K**, the closed-loop responses are given in (8) and (9).

We now prove $(ii) \Rightarrow (i)$. Suppose we have

$$\left(egin{bmatrix} oldsymbol{\delta}_x \ oldsymbol{\delta}_y \end{bmatrix}
ightarrow egin{bmatrix} \mathbf{x} \ \mathbf{u} \end{bmatrix}
ight) = egin{bmatrix} oldsymbol{\Phi}_{xx} & oldsymbol{\Phi}_{xy} \ oldsymbol{\Phi}_{ux} & oldsymbol{\Phi}_{uy} \end{bmatrix} \in \mathcal{F}_T.$$

From (9), it is not difficult to check $\Phi_{xu} = \Phi_{xx}B \in \mathcal{F}_T$, and

$$\begin{split} & \boldsymbol{\Phi}_{yx} = C\boldsymbol{\Phi}_{xx} \in \mathcal{F}_T, & \boldsymbol{\Phi}_{yy} = C\boldsymbol{\Phi}_{xy} + I \in \mathcal{F}_T \\ & \boldsymbol{\Phi}_{yu} = C\boldsymbol{\Phi}_{xx}B \in \mathcal{F}_T, & \boldsymbol{\Phi}_{uu} = \boldsymbol{\Phi}_{ux}B + I \in \mathcal{F}_T. \end{split}$$

This means that the statement (i) is true. Similar arguments can prove $(iii) \Rightarrow (v)$ and $(iv) \Rightarrow (v)$.

Finally, if (A, B, C) and (A_k, B_k, C_k) are both controllable and observable, we prove that $(v) \Rightarrow (i)$. According to (17), we have the following state-space realization

$$\begin{pmatrix} \begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \end{pmatrix} = \hat{C}_1 (zI - A_{\rm cl})^{-1} \hat{B}_2 + \begin{bmatrix} I & 0 \\ D_k & I \end{bmatrix}, \quad (28)$$

where \hat{C}_1 and \hat{B}_2 are defined in (18). In Appendix C, we show that $(A_{\rm cl}, \hat{B}_2, \hat{C}_1)$ is controllable and observable. This means that the eigenvalues of $A_{\rm cl}$ are the same as the poles of the transfer matrices [1, Chapter 3]. Therefore, if the statement (v) is true, then the closed-loop matrix $A_{\rm cl}$ only has zero eigenvalues and no eigenvalues of $A_{\rm cl}$ is hidden from the input-output behavior. This completes the proof.

Upon defining the following sets

$$C_{\text{SLP}} = \{ \mathbf{K} = \mathbf{\Phi}_{uy} - \mathbf{\Phi}_{ux} \mathbf{\Phi}_{xx}^{-1} \mathbf{\Phi}_{xy} \mid \mathbf{\Phi}_{xx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy} \in \mathcal{F}_T \text{ are in the affine subspace (21)} \},$$

$$C_{XY} = \{ \mathbf{K} - \mathbf{\Phi}_{xx} \mathbf{\Phi}_{xx}^{-1} \mid \mathbf{\Phi}_{xx} \mathbf{\Phi}_{xx} \mathbf{\Phi}_{xx} \in \mathcal{F}_T \}$$

$$C_{M1} = \{ \mathbf{K} = \mathbf{\Phi}_{uy} \mathbf{\Phi}_{yy}^{-1} \mid \mathbf{\Phi}_{yx}, \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{uy} \in \mathcal{F}_T$$
 are in the affine subspace (25)},

$$C_{M2} = \{ \mathbf{K} = \mathbf{\Phi}_{uu}^{-1} \mathbf{\Phi}_{uy} \mid \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xu}, \mathbf{\Phi}_{uu} \in \mathcal{F}_T$$
 are in the affine subspace (27)},

$$C_{\text{IOP}} = \{ \mathbf{K} = \mathbf{\Phi}_{uy} \mathbf{\Phi}_{yy}^{-1} \mid \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{yu}, \mathbf{\Phi}_{uu} \in \mathcal{F}_T$$
 are in the affine subspace (23)},

it is easy to derive the following corollary.

Corollary 5 If (A, B, C) and (A_k, B_k, C_k) are both stabilizable and detectable, we have $C_{SLP} \subseteq C_{M1} \subseteq C_{IOP} \subseteq C_{stab}$ and $C_{SLP} \subseteq C_{M2} \subseteq C_{IOP} \subseteq C_{stab}$. If (A, B, C) and (A_k, B_k, C_k) are both controllable and observable, we have $C_{SLP} = C_{M1} = C_{M2} = C_{IOP} \subseteq C_{stab}$.

Theoretically, the closed-loop parameterizations in Propositions 1-4 are equivalent to each other. However, after imposing the FIR approximation on the decision variables, Corollary 5 shows that the IOP [7] in Proposition 2 has the best ability to approximate the set of stabilizing controllers $\mathcal{C}_{\text{stab}}$, as it exclusively deals with the maps from inputs to outputs without touching the system state. Precisely, when there are some stable uncontrollable and/or unobservable modes in (1), these modes cannot be changed by any feedback controllers and will be reflected in the closed-loop responses involving the state x. Therefore, for systems with stable uncontrollable and/or unobservable modes, the parameters in the SLP [6], or the new parameterization in Proposition 3/4 (Mixed I/II), cannot be made FIR by definition, since these parameterizations involve the state **x** and/or the disturbance on the state $\boldsymbol{\delta}_x$ explicitly. For example, consider an LTI system (1) with matrices as

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

There is one uncontrollable and unobservable mode z = 0.5, and this mode is stable. The affine constraints (21), (25), (27) are all infeasible for any FIR approximation with

³ Depending on the choice of the cost function, optimal controller synthesis may be cast as a quadratic program (QP) under the FIR assumption; see a case study in Section 7.

finite horizon T, while the IOP in Proposition 2 is feasible as long as the horizon $T \geq 1$.

Remark 3 Note that if there are some stable uncontrollable and/or unobservable modes in (1), one may perform a model reduction to get an equivalent state-space realization that is controllable and observable. Then, all the closed-loop parameterizations in Propositions 1-4 have the same ability for approximating C_{stab} when imposing the FIR assumption. We note that model reduction generally destroys the underlying sparsity structure in the original system (1), which may be unfavourable for distributed controller synthesis [28].

5.2 Controller implementation via state-space realization

In Propositions 1–4, to get the controller \mathbf{K} , we need to compute the inverse of some transfer matrix as well as the product of transfer matrices. For the SLP in [6], the authors proposed the following implementation of the controller $\mathbf{K} = \mathbf{\Phi}_{uy} - \mathbf{\Phi}_{ux} \mathbf{\Phi}_{xx}^{-1} \mathbf{\Phi}_{xy}$ from the system responses matrices $\mathbf{\Phi}_{xx}, \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{uy}$:

$$z\beta = z(I - z\mathbf{\Phi}_{xx})\beta - z\mathbf{\Phi}_{xy}\mathbf{y},$$

$$\mathbf{u} = z\mathbf{\Phi}_{ux}\beta + \mathbf{\Phi}_{uy}\mathbf{y}.$$
(29)

The implementation (29) avoids the explicit computation of matrix inverse and matrix product. However, the controller matrices in (29) still contain transfer matrices. Motivated by [25], this subsection provides a standard state-space realization (3) for the controller in closed-loop parameterizations after imposing the FIR approximation.

We first consider the controller $\mathbf{K} = \mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1}$ in Proposition 2 (IOP) and Proposition 3 (Mixed I). We assume that the system response $\mathbf{\Phi}_{uy}$ and $\mathbf{\Phi}_{yy}$ are FIR transfer matrices of horizon T, *i.e.*,

$$\mathbf{\Phi}_{uy} = \sum_{t=0}^{T} U_t \frac{1}{z^t} \in \mathcal{RH}_{\infty}, \ \mathbf{\Phi}_{yy} = \sum_{t=0}^{T} Y_t \frac{1}{z^t} \in \mathcal{RH}_{\infty}.$$
 (30)

Upon defining the following real matrices

$$\hat{U} = \begin{bmatrix} U_1 & U_2 & \dots & U_T \end{bmatrix} \in \mathbb{R}^{m \times pT},
\hat{Y} = \begin{bmatrix} Y_1 & Y_2 & \dots & Y_T \end{bmatrix} \in \mathbb{R}^{p \times pT},$$
(31)

and $Z_p \in \mathbb{R}^{pT \times pT}$ as the down shift operator with sub-diagonal containing identity matrices of dimension $p \times p$ (see Appendix D for an explicit form), and $\mathcal{I}_p = [I_p, 0, \dots, 0]^\mathsf{T} \in \mathbb{R}^{pT \times p}$, we have the following result.

Theorem 3 Suppose that Φ_{uy} and Φ_{yy} are FIR transfer matrices with horizon T in (30). A state-space realization for the output feedback controller $\mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1}$ is given by

$$\mathbf{K} = \left[\frac{Z_p - \mathcal{I}_p \hat{Y} \left| -\mathcal{I}_p}{U_0 \hat{Y} - \hat{U} \right| U_0} \right]. \tag{32}$$

A state-space realization for the controller $\mathbf{K} = \mathbf{\Phi}_{uu}^{-1} \mathbf{\Phi}_{uy}$ in Proposition 4 (Mixed II) can be developed similarly.

For the SLP controller $\mathbf{K} = \mathbf{\Phi}_{uy} - \mathbf{\Phi}_{ux}\mathbf{\Phi}_{xx}^{-1}\mathbf{\Phi}_{xy}$ in Proposition 1, we assume the system responses $\mathbf{\Phi}_{uy}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{xx}, \mathbf{\Phi}_{xy}$ are FIR transfer matrices of horizon

T, i.e.,

$$\Phi_{ux} = \sum_{t=0}^{T} M_t \frac{1}{z^t} \in \mathcal{RH}_{\infty}, \quad \Phi_{xx} = \sum_{t=0}^{T} R_t \frac{1}{z^t} \in \mathcal{RH}_{\infty}, \\
\Phi_{xy} = \sum_{t=0}^{T} N_t \frac{1}{z^t} \in \mathcal{RH}_{\infty}.$$
(33)

Upon defining the following matrices

$$\hat{M} = \begin{bmatrix} M_2 & M_3 & \dots & M_T \end{bmatrix} \in \mathbb{R}^{m \times n\hat{T}},
\hat{R} = \begin{bmatrix} R_2 & R_3 & \dots & R_T \end{bmatrix} \in \mathbb{R}^{n \times n\hat{T}},
\hat{N} = \begin{bmatrix} N_1 & N_2 & \dots & N_T \end{bmatrix} \in \mathbb{R}^{n \times pT},$$
(34)

and $Z_n \in \mathbb{R}^{n\hat{T} \times n\hat{T}}$ as the down shift operator with subdiagonal containing identity matrices of dimension $n \times n$, and $\mathcal{I}_n = [I_n, 0, \dots, 0]^\mathsf{T} \in \mathbb{R}^{n\hat{T} \times n}$, with $\hat{T} = T - 1$, we have the following result.

Theorem 4 Suppose that Φ_{uy} , Φ_{ux} , Φ_{xx} , Φ_{xy} are FIR transfer matrices with horizon T in (30) and (33). A state-space realization for the output feedback controller $\mathbf{K} = \Phi_{uy} - \Phi_{ux}\Phi_{xx}^{-1}\Phi_{xy}$ is given by

$$\mathbf{K} = \begin{bmatrix} Z_n - \mathcal{I}_n \hat{R} & -\mathcal{I}_n \hat{N} & 0\\ 0 & Z_p & \mathcal{I}_p\\ \hat{M} - M_1 \hat{R} & \hat{U} - M_1 \hat{N} & U_0 \end{bmatrix}.$$
(35)

The proofs of Theorems 3 and 4 are motivated by [25], and are based on some standard operations on dynamic systems (see, e.g., [1, Chapter 3.6]). We provide the proofs in Appendix D for completeness.

6 Numerical robustness of closed-loop parameterizations

The previous sections highlighted the benefits of closed-loop parameterizations: the set of internally stabilizing controllers can be fully characterized by a set of affine constraints on certain closed-loop responses, leading to finite-dimensional convex optimization problems for controller synthesis after imposing the FIR constraints. However, numerical solutions computed via floating point (even with arbitrary precision) can never be exact for the affine constraints, especially considering the finite stopping criteria used in common solvers, like SeDuMi [29] and Mosek [30]. Therefore, there always exist certain numerical errors for the affine constraints in Propositions 1–4.

This section discusses the numerical robustness of closed-loop parameterizations, and investigates how the numerical errors in the affine constraints affect the stability of the closed-loop system. We highlight that the Youla parameterization (7) can eliminate the affine constraints in Propositions 1–4 explicitly. An overview of the results in this section is presented in Table 1.

6.1 Robustness results for the IOP and the SLP

We begin with the IOP in Proposition 2. The transfer matrices $\hat{\Phi}_{yy}$, $\hat{\Phi}_{uy}$, $\hat{\Phi}_{uy}$, $\hat{\Phi}_{uu}$ only approximately satisfy the

Table 1 Comparison of numerical robustness among different closed-loop parameterizations

	Coprime factorization	Equality constraints	Controller recovery K	Open-loop stable plants	Open-loop unstable plants	Pre-stabilizing the plant ²
SLP [6]	No	Yes	$egin{aligned} oldsymbol{\Phi}_{uy} - oldsymbol{\Phi}_{ux} oldsymbol{\Phi}_{xy}^{-1} oldsymbol{\Phi}_{xy} \ oldsymbol{\Phi}_{uy} (I + C oldsymbol{\Phi}_{xy})^{-1} \end{aligned}$	*	* X	*
			$\Phi_{ux}\Phi_{xx}^{-1}$ (when $C=I$) ¹	· ✓	√	✓
IOP [7]	No	Yes	$oldsymbol{\Phi}_{uy}oldsymbol{\Phi}_{uy}^{-1}$	\checkmark	X	\checkmark
Mixed I	No	Yes	$\mathbf{\Phi}_{uy}\mathbf{\Phi}_{uy}^{-1}$	\checkmark	X	\checkmark
Mixed II	No	Yes	$\boldsymbol{\Phi}_{uu}^{-1}\boldsymbol{\Phi}_{uy}$	\checkmark	X	✓
Youla [4]	Yes	No	(7)	\checkmark	\checkmark	\checkmark

 $[\]overline{}^{1}$: This only works for the state feedback case, *i.e.*, C = I.

affine constraint (23), i.e., we have

$$\begin{bmatrix} I - \mathbf{G} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{yy} & \hat{\mathbf{\Phi}}_{yu} \\ \hat{\mathbf{\Phi}}_{uy} & \hat{\mathbf{\Phi}}_{uu} \end{bmatrix} = \begin{bmatrix} I + \mathbf{\Delta}_1 & \mathbf{\Delta}_2 \end{bmatrix},$$

$$\begin{bmatrix} \hat{\mathbf{\Phi}}_{yy} & \hat{\mathbf{\Phi}}_{yu} \\ \hat{\mathbf{\Phi}}_{uy} & \hat{\mathbf{\Phi}}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_3 \\ I + \mathbf{\Delta}_4 \end{bmatrix},$$

$$\hat{\mathbf{\Phi}}_{yy}, \hat{\mathbf{\Phi}}_{uy}, \hat{\mathbf{\Phi}}_{yu}, \hat{\mathbf{\Phi}}_{uu} \in \mathcal{RH}_{\infty},$$

$$(36)$$

where the residuals are

$$\begin{split} & \boldsymbol{\Delta}_1 = \hat{\boldsymbol{\Phi}}_{yy} - \mathbf{G}\hat{\boldsymbol{\Phi}}_{uy} - I, & \boldsymbol{\Delta}_2 = \hat{\boldsymbol{\Phi}}_{yu} - \mathbf{G}\hat{\boldsymbol{\Phi}}_{uu}, \\ & \boldsymbol{\Delta}_3 = -\hat{\boldsymbol{\Phi}}_{yy}\mathbf{G} + \hat{\boldsymbol{\Phi}}_{yu}, & \boldsymbol{\Delta}_4 = -\hat{\boldsymbol{\Phi}}_{uy}\mathbf{G} + \hat{\boldsymbol{\Phi}}_{uu} - I. \end{split}$$

Theorem 5 Let $\hat{\Phi}_{yy}$, $\hat{\Phi}_{uy}$, $\hat{\Phi}_{yu}$, $\hat{\Phi}_{uu}$ satisfy (36). Then, we have the following statements.

- (1) In the case of $\mathbf{G} \in \mathcal{RH}_{\infty}$, the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}\hat{\mathbf{\Phi}}_{yy}^{-1}$ internally stabilizes the plant \mathbf{G} if and only if $(I + \mathbf{\Delta}_1)^{-1}$ is stable.
- (2) In the case of $\mathbf{G} \notin \mathcal{RH}_{\infty}$, the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}\hat{\mathbf{\Phi}}_{yy}^{-1}$ cannot guarantee the internal stability of the closed-loop system if there exist non-zero residuals $\mathbf{\Delta}_i$.

Proof: Given a controller \mathbf{K} , the closed-loop responses from $(\boldsymbol{\delta}_y, \boldsymbol{\delta}_u)$ to (\mathbf{y}, \mathbf{u}) are

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{G}\mathbf{K})^{-1} & (I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} \\ \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} & I + \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix}.$$

Considering $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}\hat{\mathbf{\Phi}}_{yy}^{-1}$, where $\hat{\mathbf{\Phi}}_{yy}, \hat{\mathbf{\Phi}}_{uy}, \hat{\mathbf{\Phi}}_{yu}, \hat{\mathbf{\Phi}}_{uu}$ satisfy (36), we can verify the following identities:

$$(I - \mathbf{G}\mathbf{K})^{-1} = (I - \mathbf{G}\hat{\mathbf{\Phi}}_{uy}\hat{\mathbf{\Phi}}_{yy}^{-1})^{-1}$$

$$= \hat{\mathbf{\Phi}}_{yy}(I + \mathbf{\Delta}_1)^{-1},$$

$$(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = \hat{\mathbf{\Phi}}_{yy}(I + \mathbf{\Delta}_1)^{-1}\mathbf{G},$$

$$\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} = \hat{\mathbf{\Phi}}_{uy}(I + \mathbf{\Delta}_1)^{-1},$$
(37)

and
$$I + \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = I + \hat{\mathbf{\Phi}}_{uy}(I + \mathbf{\Delta}_1)^{-1}\mathbf{G}$$
.

Proof of Statement 1: Suppose that $\mathbf{G} \in \mathcal{RH}_{\infty}$. If (I +

 $(\Delta_1)^{-1}$ is stable, it is easy to verify that all transfer matrices in (37) are stable. This means that

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_{\infty}.$$

By Theorem 1, we know $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy} \hat{\mathbf{\Phi}}_{yy}^{-1}$ internally stabilizes the plant \mathbf{G} . If $(I + \mathbf{\Delta}_1)^{-1}$ is unstable, then the closed-loop response from $\boldsymbol{\delta}_y$ to \mathbf{y} will be unstable in general, and thus the controller does not internally stabilize the system.

Proof of Statement 2: If **G** is unstable, the transfer matrices in (37) cannot be guaranteed to be stable if $\Delta_1 \neq 0$. When $\Delta_1 = 0$, we have $(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = \hat{\Phi}_{yy}\mathbf{G} = \hat{\Phi}_{yu} - \Delta_3$

$$I + \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = I + \hat{\mathbf{\Phi}}_{uy}\mathbf{G} = \hat{\mathbf{\Phi}}_{uu} - \mathbf{\Delta}_4.$$

Note that the residuals Δ_3 , Δ_4 in (36) can be unstable if **G** is unstable (since the product of an unstable transfer matrix and a stable one can be unstable). Thus, the controller cannot guarantee the internal stability of the closed-loop system unless $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$, $\Delta_4 = 0$.

We now focus on the SLP in Proposition 1. The transfer matrices $\hat{\Phi}_{xx}$, $\hat{\Phi}_{ux}$, $\hat{\Phi}_{xy}$, $\hat{\Phi}_{uy}$ only approximately satisfy the affine constraint (21), *i.e.*, we have

$$\begin{bmatrix} zI - A - B \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{xx} & \hat{\mathbf{\Phi}}_{xy} \\ \hat{\mathbf{\Phi}}_{ux} & \hat{\mathbf{\Phi}}_{uy} \end{bmatrix} = \begin{bmatrix} I + \hat{\boldsymbol{\Delta}}_{1} & \hat{\boldsymbol{\Delta}}_{2} \end{bmatrix},$$

$$\begin{bmatrix} \hat{\mathbf{\Phi}}_{xx} & \hat{\mathbf{\Phi}}_{xy} \\ \hat{\mathbf{\Phi}}_{ux} & \hat{\mathbf{\Phi}}_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = \begin{bmatrix} I + \hat{\boldsymbol{\Delta}}_{3} \\ \hat{\boldsymbol{\Delta}}_{4} \end{bmatrix},$$

$$\hat{\mathbf{\Phi}}_{xx}, \hat{\mathbf{\Phi}}_{ux}, \hat{\mathbf{\Phi}}_{xy}, \hat{\mathbf{\Phi}}_{uy} \in \mathcal{RH}_{\infty},$$

$$(38)$$

where the residuals are

$$\hat{\Delta}_1 = (zI - A)\hat{\Phi}_{xx} - B\hat{\Phi}_{ux} - I,$$

$$\hat{\Delta}_2 = (zI - A)\hat{\Phi}_{xy} - B\hat{\Phi}_{uy},$$

$$\hat{\Delta}_3 = \hat{\Phi}_{xx}(zI - A) - \hat{\Phi}_{xy}C - I$$

and
$$\hat{\Delta}_4 = \hat{\Phi}_{ux}(zI - A) - \hat{\Phi}_{uy}C$$
.

 $^{^{2}}$: This applies an initial stabilizing controller that is stable itself (see Proposition 5).

^{*:} The situation requires care-by-case analysis; see Theorem 6 and Section 6.2 for details.

 $[\]checkmark$: The parameterization is numerically robust (see Corollary 6).

X: The parameterization cannot guarantee the closed-loop stability if small numerical mismatches in the equality constraints exist.

Note that there are multiple ways to recover the controller \mathbf{K} in the SLP framework. The SLP controller can also be recovered in another way as $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1}$ [6]. Now, we have the following result.

Theorem 6 Let $\hat{\Phi}_{xx}$, $\hat{\Phi}_{ux}$, $\hat{\Phi}_{xy}$, $\hat{\Phi}_{uy}$ satisfy (38). We have the following statements.

- (1) In the state feedback case, i.e., C = I, the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{ux}\hat{\mathbf{\Phi}}_{xx}^{-1}$ internally stabilizes the plant \mathbf{G} if and only if $(I + \hat{\mathbf{\Delta}}_1)^{-1}$ is stable.
- (2) The four-block SLP controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy} \hat{\mathbf{\Phi}}_{ux} \hat{\mathbf{\Phi}}_{xx}^{-1} \hat{\mathbf{\Phi}}_{xy}$ cannot guarantee the internal stability of the closed-loop system if $(I + \hat{\boldsymbol{\Delta}})^{-1}$ is unstable, where ⁴

$$\hat{\boldsymbol{\Delta}} := \hat{\boldsymbol{\Delta}}_3 + \hat{\boldsymbol{\Phi}}_{xx} (I + \hat{\boldsymbol{\Delta}}_1)^{-1} (B\hat{\boldsymbol{\Delta}}_4 - (zI - A)\hat{\boldsymbol{\Delta}}_3). \tag{39}$$

(3) For the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1}$,

a) if $G \in \mathcal{RH}_{\infty}$, K internally stabilizes the plant G if and only if $(I + C(zI - A)^{-1}\hat{\Delta}_2)^{-1}$ is stable.

b) if $\mathbf{G} \notin \mathcal{RH}_{\infty}$, \mathbf{K} cannot guarantee the internal stability of the closed-loop system if there exist nonzero residuals $\hat{\Delta}_i$, i = 1, ..., 4.

Proof: The proof of Statement 1 is presented in [24, Theorem 4.3]. We prove the second statement here. Given $\hat{\Phi}_{xx}$, $\hat{\Phi}_{ux}$, $\hat{\Phi}_{xy}$, $\hat{\Phi}_{uy}$ satisfying (38) and the controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$, we consider the closed-loop response from $\boldsymbol{\delta}_x$ to \mathbf{x} . After some tedious algebra (see Appendix E), we derive

$$(zI - A - B\mathbf{K}C)^{-1} = (I + \hat{\Delta})^{-1}\hat{\Phi}_{xx}(I + \hat{\Delta}_1)^{-1},$$
(40)

with $\hat{\Delta}$ defined in (39). If $(I + \hat{\Delta})^{-1}$ is unstable, there is no guarantee that the closed-loop response from δ_x to \mathbf{x} is stable. In this case, the controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$ cannot internally stabilize the plant.

For Statement 3, considering Corollary 1, we only need to check the closed-loop response from δ_u to \mathbf{u} , which is

$$\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}$$

$$= \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1}(I - \mathbf{G}\hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1})^{-1}$$

$$= \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy} - C(zI - A)^{-1}B\hat{\mathbf{\Phi}}_{uy})^{-1}$$

$$= \hat{\mathbf{\Phi}}_{uv}(I + C(zI - A)^{-1}\hat{\mathbf{\Delta}}_{2})^{-1}.$$

The rest of the proof is similar to Theorem 5.

Similar robustness results can be derived for the Mixed I/II parameterizations (Propositions 3/4); see Appendix F for details. Theorems 5 and 6 can now be combined with the small gain theorem [1, Theorem 9.1] to provide simple sufficient conditions for numerical robustness.

Corollary 6 Let $\hat{\Phi}_{yy}$, $\hat{\Phi}_{uy}$, $\hat{\Phi}_{yu}$, $\hat{\Phi}_{uu}$ satisfy (36). Then

– for open-loop stable plants, the IOP controller $\mathbf{K} = \hat{\Phi}_{uy}\hat{\Phi}_{yy}^{-1}$ internally stabilizes the plant if $\|\mathbf{\Delta}_1\|_{\infty} < 1$.

Let $\hat{\Phi}_{xx}$, $\hat{\Phi}_{ux}$, $\hat{\Phi}_{xy}$, $\hat{\Phi}_{uy}$ satisfy (38). Then

– for the state feedback case, the SLP controller $\mathbf{K} = \mathbf{K}$

 $\hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}$ internally stabilizes the plant if $\|\hat{\Delta}_1\|_{\infty} < 1$.

– for open-loop stable plants, the SLP controller $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ internally stabilizes the plant if

$$\|\hat{\Delta}_2\|_{\infty} \le \frac{1}{\|C(zI - A)\|_{\infty}}.$$

The sufficient condition for robustness of the SLP state feedback case first appeared in [31], which is one key result in the recent learning-based control applications [15, 16].

Remark 4 The controller recovery in IOP/Mixed I/II, the state-feedback SLP controller, and the SLP controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1}$ only involve two parameters explicitly. Thus, their robustness analysis is more straightforward compared to the four-block SLP controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy} - \hat{\mathbf{\Phi}}_{ux}\hat{\mathbf{\Phi}}_{xx}^{-1}\hat{\mathbf{\Phi}}_{xy}$. As shown in Theorem 6, the residuals $\hat{\Delta}_i$, $i=1,\ldots,4$ play a more complex role in the resulting closed-loop responses, irrespective of state- or output-feedback, or open-loop stability of the plant. Since the closed-loop responses from $\boldsymbol{\delta}_x, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u$ to $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}$ can be computed using these residuals $\hat{\Delta}_i$, one may find sophisticated sufficient conditions on the residuals $\hat{\Delta}_i$ to ensure the internally stability of the closed-loop system. These conditions might relate these residuals with numerical solutions such as Φ_{xx} . Deriving such conditions and finding tractable ways to enforce these conditions are beyond the scope of this paper and left as future work.

6.2 Implications in numerical computation

Here, we discuss the implication of Theorems 5, 6 and Corollary 6 for numerical computation. In practice, the computational residuals $\mathbf{\Delta}_1 = \hat{\mathbf{\Phi}}_{yy} - \mathbf{G}\hat{\mathbf{\Phi}}_{uy} - I$ (when **G** is stable) and $\hat{\mathbf{\Delta}}_1 = (zI - A)\hat{\mathbf{\Phi}}_{xx} - B\hat{\mathbf{\Phi}}_{xy} - I$ are very small numerically. It is fairly safe to say that $\|\mathbf{\Delta}_1\|_{\infty} < 1$ and $\|\hat{\mathbf{\Delta}}_1\|_{\infty} < 1$ in numerical computation using any common interior-point solvers, such as SeDuMi [29] and Mosek [30]. Similar statements are true for the Mixed I/II parameterizations (see Appendix F). This observation leads to the following summary (see Table 1 for an overview).

Numerical Robustness. Consider closed-loop parameterizations (SLP, IOP, Mixed I/II) in numerical computation. We have

- (i) the SLP with controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{ux}\hat{\mathbf{\Phi}}_{xx}^{-1}$ is numerically robust in the state feedback case;
- (ii) the IOP, Mixed I/II, and SLP with controller $\hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ are numerically robust for open-loop stable plants.

On the other hand, we have

- (I) the IOP, Mixed I/II, and SLP with controller $\hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ are not numerically robust for open-loop unstable plants, irrespective of having state- or output-feedback;
- (II) the SLP with controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy} \hat{\mathbf{\Phi}}_{ux}\hat{\mathbf{\Phi}}_{xx}^{-1}\hat{\mathbf{\Phi}}_{xy}$ is not numerically robust in general, irrespective of open-loop stability of the plant.

The statements (i), (ii) and (I) are easy to see from the previous section. The statement (II) comes form Theorem 6 but requires more attention. Although the computational

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⁴ Note that because $\|\hat{\Delta}\|_{\infty}$ may large than 1, there is no guarantee that $(I + \hat{\Delta})^{-1}$ is always stable. See Example (41).

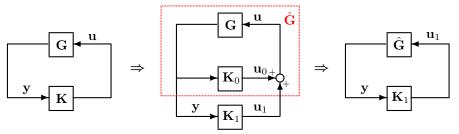


Figure 2. Given an initial controller $\mathbf{K}_0 \in \mathcal{C}_{\text{stab}} \cap \mathcal{RH}_{\infty}$, we search for \mathbf{K}_1 to stabilize the new stable plant $\hat{\mathbf{G}} := (I - \mathbf{GK}_0)^{-1}\mathbf{G}$.

residuals $\hat{\Delta}_1$, $\hat{\Delta}_2$, $\hat{\Delta}_3$, $\hat{\Delta}_4$ in (38) are typically very small element-wise by interior-point solvers, we still cannot guarantee that $\|\hat{\Delta}\|_{\infty} < 1$ (where $\hat{\Delta}$ is defined in (39)), since $\hat{\Delta}$ involves $\hat{\Phi}_{xx}$ explicitly. Consequently, it is possible that $(I + \hat{\Delta})^{-1}$ is unstable in numerical computation. Therefore, one may argue that the four-block SLP controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$ is not numerically robust in general ⁵. Further, we notice that the controller implementation (29) proposed in [6] also suffers the issue of numerical instability, as the right-hand-side of (40) represents the corresponding closed-loop response using (29).

Example 1 To understand the role of the residuals, we present a simple example. Consider a stable LTI system (1) with

$$A = 0, \quad B = 1, \quad C = 1.$$

It can be verified that the following transfer functions

$$\hat{\Phi}_{xx} = \frac{1}{z} + \frac{(z-5)(z+6)^2}{z^5},$$

$$\hat{\Phi}_{ux} = \frac{(z-5)(z+6)^2}{z^4},$$

$$\hat{\Phi}_{xy} = \frac{(z-5)(z+6)^2}{z^4} - \frac{1}{1000} \frac{(z+2)^2}{z^3},$$

$$\hat{\Phi}_{uy} = \frac{(z-5)(z+6)^2}{z^3},$$
(41)

satisfy (38) with residuals

$$\hat{\Delta}_1 = 0, \hat{\Delta}_2 = -\frac{(z+2)^2}{1000z^2}, \hat{\Delta}_3 = \frac{(z+2)^2}{1000z^3}, \hat{\Delta}_4 = 0.$$

For this example, we verify that $(I + \hat{\Delta})^{-1}$ has a pair of unstable poles $z = 0.9522 \pm 0.5226i$, despite the norm $\|\hat{\Delta}_3\|_{\infty} = 9 \times 10^{-3}$ being very small, and that this pair of unstable poles also appears in the closed-loop system $(zI - A - BKC)^{-1}$ using the controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$. This example is open-loop stable and it is also in state feedback form. Nonetheless, a small residual can destabilize the closed-loop using the four-block SLP controller.

We remark that in Example 1, since A = 0, the optimal LQR controller will be $\mathbf{K} = 0$ for any weight matrices Q and R. Thus, any sensible formulation of optimal control problems using the SLP will not lead to the highly suboptimal solution (41). However, we emphasize that the numerical residuals play a complex role in the closed-loop system

using the four-block SLP controller, and residuals with a small norm may lead to an undesirable destabilization situation. Unlike the four-block SLP controller, from Corollary 6, the state feedback SLP controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{ux}\hat{\mathbf{\Phi}}_{xx}^{-1}$ is numerically robust as long as $\|\hat{\mathbf{\Delta}}_1\|_{\infty} < 1$. Since $\hat{\mathbf{\Delta}}_1 = 0$ in Example 1, the closed-loop system $(zI - A - B\mathbf{K}C)^{-1}$ has all zero eigenvalues using $\mathbf{K} = \hat{\mathbf{\Phi}}_{ux}\hat{\mathbf{\Phi}}_{xx}^{-1}$. Meanwhile, we can verify that $\|C(zI - A)^{-1}\hat{\mathbf{\Delta}}_2\|_{\infty} = 0.009 < 1$, thus it is guaranteed that the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1}$ internally stabilizes the plant (the largest norm of the closed-loop eigenvalues is 0.1675).

The question remains whether the phenomenon highlighted in Example 1 may lead to numerical instability when solving optimal control formulation in practice. We observed several cases where the four-block SLP controller failed to stabilize the plant even using the default setting (high precision) in Mosek [30] for numerical computation 6 . This is likely due to $\|\hat{\Phi}_{xx}\|_{\infty}$ being high, despite solving an optimal control formulation. How to avoid this issue requires more investigations, which is left for future work.

6.3 Open-loop unstable plants and relation with the Youla parameterization

To characterize the set of internally stabilizing controllers $\mathcal{C}_{\mathrm{stab}}$, the closed-loop parameterizations in Proposition 1–4 can avoid computing the doubly co-prime factorization of the plant *a priori*, but they all need to impose a set of affine constraints for achievable closed-loop responses. As discussed above, any small mismatch in the additional affine constraints can make the resulting controller unimplementable when the plant is open-loop unstable (IOP, Mixed I/II), and the four-block SLP controller requires a case-by-case investigation.

For the case of open-loop unstable plants, there exists a valid remedy by pre-stabilizing the plant. Suppose that G is unstable, and that a *stable* and *stabilizing* controller K_0 is known *a priori*. We can split the control signal as

$$\mathbf{u} = \mathbf{K}_0 \mathbf{y} + \mathbf{u}_1,$$

and design \mathbf{u}_1 . This is equivalent to applying the closed-loop parameterization to the new stable plant $\hat{\mathbf{G}} := (I - \mathbf{G}\mathbf{K}_0)^{-1}\mathbf{G}$ (see Figure 2 for illustration). Upon defining

⁵ As discussed in Remark 4, sufficient conditions could exist to ensure internal stability, and they will depend on $\hat{\Phi}_{xx}$.

Get the examples at https://github.com/zhengy09/h2_clp, where the system matrices $A \in \mathbb{R}^{3\times 3}, B \in \mathbb{R}^{3\times 1}, C \in \mathbb{R}^{1\times 3}$ have integer elements randomly generated from -5 to 5, and the weight matrices are chosen Q = I, R = I; see (44).

$$\hat{\mathcal{C}}_{\mathrm{stab}} := \{ \mathbf{K}_0 + \mathbf{K}_1 \mid \mathbf{K}_1 \text{ internally stabilizes } \hat{\mathbf{G}} \},$$

we have the following result.

Proposition 5 Given an initial controller $\mathbf{K}_0 \in \mathcal{C}_{stab} \cap \mathcal{RH}_{\infty}$, we have $\mathcal{C}_{stab} = \hat{\mathcal{C}}_{stab}$.

The proof is based on algebra verification; see Appendix G. Proposition 5 shows that searching over $\hat{\mathcal{C}}_{\text{stab}}$ has no conservatism. The new plant $\hat{\mathbf{G}} = (I - \mathbf{G}\mathbf{K}_0)^{-1}\mathbf{G}$ is stable, and thus any closed-loop parameterization in Propositions 1–4 for this plant has good numerical robustness ⁷. As shown in [9, Theorem 17], giving $\mathbf{K}_0 \in \mathcal{C}_{\text{stab}} \cap \mathcal{RH}_{\infty}$, the Youla parameterization (7) has a simple form as well, since one can choose an explicit doubly-coprime factorization for \mathbf{G} as

$$\mathbf{M}_{l} = (I - \mathbf{G}\mathbf{K}_{0})^{-1}, \qquad \mathbf{M}_{r} = -(I - \mathbf{K}_{0}\mathbf{G})^{-1}$$

$$\mathbf{N}_{l} = \mathbf{G}(I - \mathbf{K}_{0}\mathbf{G})^{-1}, \qquad \mathbf{N}_{r} = -\mathbf{G}(I - \mathbf{K}_{0}\mathbf{G})^{-1},$$

$$\mathbf{U}_{l} = -I, \quad \mathbf{V}_{l} = -\mathbf{K}_{0}, \quad \mathbf{U}_{r} = I, \quad \mathbf{V}_{r} = \mathbf{K}_{0}.$$

If the plant is open-loop stable, we can choose $\mathbf{K}_0 = 0$.

Unlike the closed-loop parameterizations in Propositions 1–4, the Youla parameterization (7) allows the parameter \mathbf{Q} to be freely chosen in \mathcal{RH}_{∞} with no equality constraints. Indeed, any doubly-coprime factorization of the plant can be used to eliminate the affine constraints in Propositions 1–4 exactly, as shown below.

Proposition 6 Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of \mathbf{G} . For any $\mathbf{Q} \in \mathcal{RH}_{\infty}$, the following transfer matrices

$$\Phi_{yy} = (\mathbf{U}_r - \mathbf{N}_r \mathbf{Q}) \mathbf{M}_l,
\Phi_{uy} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l,
\Phi_{yu} = (\mathbf{U}_r - \mathbf{N}_r \mathbf{Q}) \mathbf{N}_l,
\Phi_{uu} = I + (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{N}_l,$$
(42)

and

$$\Phi_{xx} = (zI - A)^{-1} + (zI - A)^{-1}B\Phi_{uy}C(zI - A)^{-1},
\Phi_{ux} = \Phi_{uy}C(zI - A)^{-1},
\Phi_{xy} = (zI - A)^{-1}B\Phi_{uy},
\Phi_{xu} = (zI - A)^{-1}B\Phi_{uu},
\Phi_{yx} = \Phi_{uu}C(zI - A)^{-1},$$
(43)

satisfy the affine constraints (21), (23), (25), (27).

The proof is based on direct verification, which is omitted here; see [23] for further discussions on the equivalence of the Youla parameterization, the IOP, and the SLP. We note that a doubly-coprime factorization can be found in the state-space domain [26], and this pre-process might introduce numerical issues that affect closed-loop stability, which is beyond the scope of this paper.

7 Case studies

In this section, we present a case study of optimal controller synthesis for *open-loop stable* plants using Propositions 1-4. We show that the optimal controller synthesis

problem can be cast into a quadratic program (QP) after imposing the FIR constraint. 8

7.1 Application to optimal controller synthesis

We consider the following optimal control problem

$$\min_{\mathbf{K}} \quad \lim_{T \to \infty} \mathbb{E} \left[\frac{1}{T} \sum_{t=0}^{T} \left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t \right) \right]
\text{s.t.} \quad x[t+1] = Ax[t] + B(u[t] + \delta_u[t]),
y[t] = Cx[t] + \delta_y[t],
\mathbf{u} = \mathbf{K} \mathbf{y},$$
(44)

where $\delta_u[t] \sim \mathcal{N}(0, I), \delta_y[t] \sim \mathcal{N}(0, I), Q \succ 0$ and $R \succ$ are performance-weight matrices with compatible dimensions. Problem (44) can be reformulated into a problem in the frequency domain

$$\min_{\mathbf{K}} \quad \left\| \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} (I - \mathbf{G}\mathbf{K})^{-1} & (I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} \\ \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} & (I - \mathbf{K}\mathbf{G})^{-1} \end{bmatrix} \right\|_{\mathcal{H}_{2}}^{2}$$
s.t. $\mathbf{K} \in \mathcal{C}_{\text{stab}}$,
$$\mathbf{K} \in \mathcal{C}_{\text{stab}}, \qquad (45)$$

where $\mathbf{G} = C(zI - A)^{-1}B$.

It is easy to see that the optimal synthesis problem (45) is non-convex in terms of \mathbf{K} since both the cost function and constraint are non-convex. Using a change of variables, as suggested in Propositions 1-4, it is equivalent to replace $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ with the affine constraints (21), (23), (25), or (27). It remains to reformulate the cost function in terms of these new variables. Simple algebra shows that

$$(I - \mathbf{GK})^{-1} = \mathbf{\Phi}_{yy} = C\mathbf{\Phi}_{xy} + I,$$

$$\mathbf{K}(I - \mathbf{GK})^{-1} = \mathbf{\Phi}_{uy},$$

$$(I - \mathbf{KG})^{-1} = \mathbf{\Phi}_{uu} = \mathbf{\Phi}_{ux}B + I,$$

and
$$(I - \mathbf{GK})^{-1}\mathbf{G} = \mathbf{\Phi}_{yu} = C\mathbf{\Phi}_{xx}B = C\mathbf{\Phi}_{xu} = \mathbf{\Phi}_{yx}B.$$

Therefore, problem (45) is equivalent to any of the following convex optimization problems (46)-(49) corresponding to Propositions 1–4, respectively.

min
$$\left\| \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} C\mathbf{\Phi}_{xy} + I & C\mathbf{\Phi}_{xx}B \\ \Phi_{uy} & \mathbf{\Phi}_{ux}B + I \end{bmatrix} \right\|_{\mathcal{H}_2}^2$$
 (46)
s.t.
$$\mathbf{\Phi}_{xx}, \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{uy} \text{ satisfy (21)}.$$

min
$$\left\| \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} \ \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} \ \mathbf{\Phi}_{uu} \end{bmatrix} \right\|_{\mathcal{H}_2}^2$$
 (47)

s.t. $\Phi_{yy}, \Phi_{yu}, \Phi_{uy}, \Phi_{uu}$ satisfy (23).

⁷ For the SLP, we use the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}(I + C\hat{\mathbf{\Phi}}_{xy})^{-1}$.

⁸ Code is available at https://github.com/zhengy09/h2_clp.

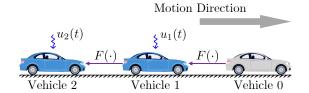


Figure 3. Each vehicle has a pre-existing car-following dynamics $F(\cdot)$ and the goal is to design an additional input $u_i(t)$, i = 1, 2 to improve the car-following performance.

min
$$\left\| \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} C\mathbf{\Phi}_{xy} + I & C\mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \right\|_{\mathcal{H}_2}^2$$
 (49)
s.t.
$$\mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xu}, \mathbf{\Phi}_{uu} \text{ satisfy (27)}.$$

Note that the \mathcal{H}_2 norm of an FIR transfer matrix $\mathbf{H} = \sum_{k=1}^{T} \frac{1}{z^k} H_k$ admits the following expression

$$\|\mathbf{H}\|_{\mathcal{H}_2}^2 = \sum_{k=1}^T \operatorname{Trace}(H_k^\mathsf{T} H_k).$$

Thus, after imposing the FIR constraint on the decision variables, problems (46)-(49) can be reformulated into QPs, for which very efficient solvers exist. After getting the solutions, the controller **K** has state-space realizations as shown in Theorems 3 and 4.

Remark 5 (Distributed optimal control) In (44), we can also impose a subspace constraint S on the controller $\mathbf{K} \in S$ to reflect local communication ability in the plant. If this subspace constraint satisfies the notion of quadratic invariance (QI) with respect to \mathbf{G} , then we have $\mathbf{K} \in S \Leftrightarrow \Phi_{uy} \in S$ [9]. Thus, distributed optimal control with QI constraints can be equivalently reformulated as (46)-(49) with an additional constraint $\Phi_{uy} \in S$. In addition to distributed optimal control, it may be desirable to introduce some spatio-temporal constraints on the system responses in (46)-(49)(see [24]), which can still be reformulated into QPs.

7.2 Numerical experiments

Here, we use a car-following control scenario [32] (see Figure 3 for illustration) to demonstrate the numerical performance of the parameterizations in Propositions 1-4.

Modelling: We denote the position and velocity of vehicle i as p_i and v_i . The spacing of vehicle i, i.e., its relative distance from vehicle i-1, is defined as $s_i = p_{i-1} - p_i, i = 1, 2$. Without loss of generality, the vehicle length is ignored. It is assumed that the leading vehicle 0 runs at a constant velocity v_0 . Each vehicle has a pre-existing car-following dynamics, and we aim to design an additional control signal $u_i(t)$ to improve the car-following performance, i.e.,

$$\dot{v}_i(t) = F(s_i(t), \dot{s}_i(t), v_i(t)) + u_i(t), \tag{50}$$

where $\dot{s}_i(t) = v_{i-1}(t) - v_i(t)$, and $F(\cdot)$ characterizes the driver's natural car-following behavior (see [33] for details). In an equilibrium car-following state, each vehicle moves with the same equilibrium velocity, i.e., $v_i(t) = v_0$, $\dot{s}_i(t) = 0$, for i = 1, 2. Assuming that each vehicle has a small perturbation from the equilibrium state (s_i^*, v^*) , we define the error state between actual and equilibrium state of vehicle

Table 2 \mathcal{H}_2 norm for different FIR lengths when solving the car-following problem.

FIR T	10	15	20	25	30	50	75
\mathcal{H}_2 norm	54.20	17.41	7.56	4.08	2.49	2.03	2.02

 $^{^{\}ddagger}$: The \mathcal{H}_2 norms from (46)-(49) have no difference up to four significant figures.

i as

$$x_i(t) = \left[\tilde{s}_i(t), \tilde{v}_i(t)\right]^\mathsf{T} = \left[s_i(t) - s_i^*, v_i(t) - v^*\right]^\mathsf{T}.$$

Applying the first-order Taylor expansion to (50), we can derive a linearized model for each vehicle (i = 1, 2)

$$\begin{cases} \dot{\tilde{s}}_{i}(t) = \tilde{v}_{i-1}(t) - \tilde{v}_{i}(t), \\ \dot{\tilde{v}}_{i}(t) = \alpha_{1}\tilde{s}_{i}(t) - \alpha_{2}\tilde{v}_{i}(t) + \alpha_{3}\tilde{v}_{i-1}(t) + u_{i}(t), \end{cases}$$

with $\alpha_1 = \frac{\partial F}{\partial s_i}$, $\alpha_2 = \frac{\partial F}{\partial \dot{s}_i} - \frac{\partial F}{\partial v_i}$, $\alpha_3 = \frac{\partial F}{\partial \dot{s}_i}$ evaluated at the equilibrium state. Assuming that we can measure the relative spacing, we arrive at the following state-space model

$$\dot{x} = \begin{bmatrix} P_1 & 0 \\ P_2 & P_1 \end{bmatrix} x + \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} (u + \delta_u),
y = \begin{bmatrix} C_1 & 0 \\ 0 & C_1 \end{bmatrix} x + \delta_y,$$
(51)

where $x = \begin{bmatrix} x_1^\mathsf{T} & x_2^\mathsf{T} \end{bmatrix}^\mathsf{T}$, $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^\mathsf{T}$, $y = \begin{bmatrix} \tilde{s}_1(t) & \tilde{s}_2(t) \end{bmatrix}^\mathsf{T}$, δ_u and δ_y are control input noise and measurement noise, respectively, and

$$P_1 = \begin{bmatrix} 0 & -1 \\ \alpha_1 & -\alpha_2 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 \\ 0 & \alpha_3 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The objective is to design $u_i(t)$ to regulate the spacing error $s_i(t) - s_i^*$ and velocity error $v_i(t) - v^*$ based on the output information y(t). This problem can be formulated into (44) in the discrete-time domain.

Numerical results: In our numerical simulations, the carfollowing parameters $\alpha_1 = 0.94$, $\alpha_2 = 1.5$, $\alpha_3 = 0.9$ are chosen according to the setup in [34], and the open-loop system is stable. Using a forward Euler-discretization of (51) with a sampling time of dT = 0.1s, we formulate the corresponding optimal controller synthesis problem (44) in discrete-time with Q = I and R = I. This can be solved via any of the convex problems (46)-(49). We varied the FIR length T from 10 to 75, and the results are listed in Table I. As expected, when increasing the FIR length, the optimal cost from (46)-(49) converges to the true value returned by the standard synthesis h2syn in MATLAB. Given an initial state $x_0 = [3, 0, -2, 0]^{\mathsf{T}}$, Figure 4 shows the time-domain responses 9 of the closed-loop system using the resulting controllers from (46)-(49) when the FIR length is T = 30. By

^{†:} The true \mathcal{H}_2 norm from h2syn in MATLAB is 2.02.

⁹ The responses from (46)-(49) have no visible difference.

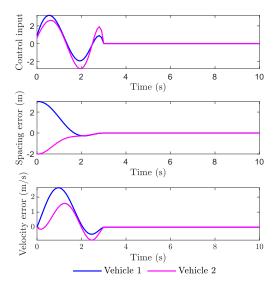


Figure 4. Responses using the controllers from (46)-(49) with FIR length T=30.

design, the closed-loop system converges to the equilibrium state within 3 seconds. Figure 4 shows the time-domain responses of the closed-loop system using the resulting controllers from (46)-(49) when the FIR length is T=75. For the same initial state, Figure 5 shows the time-domain responses of the closed-loop system when the FIR length is T=75, where the system converges the equilibrium state within 7.5 seconds with lower peak values during the transient process compared to the case T=30.

8 Conclusions

In this paper, we have characterized all possible parameterizations for the set of stabilizing controllers using closedloop maps. We have revealed two other parallel choices beyond the recent notions of SLP [6] and IOP [7]. In fact, our analysis allows to treat the SLP [6] and the IOP [7] in a unified way. After imposing the FIR approximation, the ability of the four parameterizations for encoding $\mathcal{C}_{\mathrm{stab}}$ becomes different, and the IOP enjoys the best approximation ability. These closed-loop parameterizations can avoid computing the doubly co-prime factorization of the plant a priori, but instead require imposing a set of affine constraints for achievable closed-loop responses. We have shown that any small mismatch in the additional affine constraints can make the resulting controller un-implementable when the plant is open-loop unstable (IOP, Mixed I/I) and that the output-feedback setting requires case-by-case investigation when using the SLP. Two numerically robust scenarios are the SLP in the state feedback case and the IOP for openloop stable plants. Given an initial stabilizing controller that is also stable itself, we can use this initial controller to parameterize all internally stabilizing controller and avoid the numerical instability issue.

One future direction is to address decentralized control, e.g., the notion of quadratic invariance (QI) [9] and sparsity invariance (SI) [22], using different parameterizations. Also, similar to SLP [15,16] and Youla [17–19], it will be extremely interesting to investigate the features of different parameterizations in robust synthesis for uncertain systems

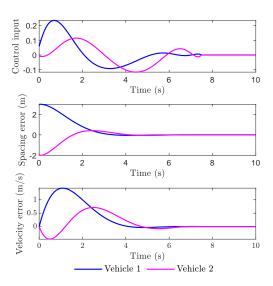


Figure 5. Responses using the controllers from (46)-(49) with FIR length T=75.

and their applications in learning-based control. Finally, we have established that closed-loop parameterizations are more subtle in practice for open-loop unstable plants with output feedback, and further investigation is needed to unravel a more precise and thorough understanding of related aspects.

Acknowledgement: The authors would like to thank Nikolai Matni, James Anderson and John C Doyle for several insightful discussions, particularly around the robustness of the SLS framework. Also, the authors thank John C Doyle for his encouragement to find a simple SISO example, eventually leading to (41).

Appendix

A Proof of stabilizability of (19)

Consider a feedback gain $K=\left[K_1\ K_2\right],$ where $K_1\in\mathbb{R}^{n\times n},K_2\in\mathbb{R}^{n\times p}.$ We have

$$\begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$
$$= \begin{bmatrix} A + BD_k + K_1 & BC_k + K_2 \\ B_k & A_k \end{bmatrix}.$$

Since (A_k, B_k) is stabilizable, there exists $F_k \in \mathbb{R}^{n \times q}$ such that $A_k + B_k F_k$ is stable. By choosing

$$K_1 = -A - BD_k - I + F_k B_k$$

$$K_2 = -BC_k - (A + BD_k + K_1)F_k + F_k(A_k + B_k F_k),$$

it can be easily verify that

$$\begin{bmatrix} I & F_k \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A + BD_k + K_1 & BC_k + K_2 \\ B_k & A_k \end{bmatrix} \begin{bmatrix} I & F_k \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} -I & 0 \\ B_k & A_k + B_k F_k \end{bmatrix}.$$

is stable. Since the similarity transformation does not change eigenvalues, there exist K_1, K_2 such that

$$\begin{bmatrix} A + BD_k + K_1 & BC_k + K_2 \\ B_k & A_k \end{bmatrix}$$

is stable. This completes the proof. \Box

B Proof of Proposition 4

Statement 1: Given $\mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$, it is easy to derive that the closed-loop responses (26) are

$$\begin{aligned} & \mathbf{\Phi}_{xy} = \mathbf{\Phi}_{xx} B \mathbf{K}, & \mathbf{\Phi}_{xu} = \mathbf{\Phi}_{xx} B, \\ & \mathbf{\Phi}_{uy} = \mathbf{K} (C \mathbf{\Phi}_{xx} B \mathbf{K} + I), & \mathbf{\Phi}_{uu} = \mathbf{K} C \mathbf{\Phi}_{xx} B + I, \end{aligned}$$

where $\Phi_{xx} = (zI - A - BKC)^{-1} \in \mathcal{RH}_{\infty}$. They are all stable by definition. Then, it is not difficult to verify that

$$(zI - A)\mathbf{\Phi}_{xy} - B\mathbf{\Phi}_{uy}$$

$$= (zI - A)\mathbf{\Phi}_{xx}B\mathbf{K} - B\mathbf{K}(C\mathbf{\Phi}_{xx}B\mathbf{K} + I)$$

$$= ((zI - A)\mathbf{\Phi}_{xx} - B\mathbf{K}C\mathbf{\Phi}_{xx} - I)B\mathbf{K} = 0,$$

and

$$(zI - A)\mathbf{\Phi}_{xu} - B\mathbf{\Phi}_{uu}$$

$$= (zI - A)\mathbf{\Phi}_{xx}B - B(C\mathbf{\Phi}_{xx}B\mathbf{K} + I)$$

$$= ((zI - A)\mathbf{\Phi}_{xx} - B\mathbf{K}C\mathbf{\Phi}_{xx} - I)B = 0,$$

and that $-\mathbf{\Phi}_{xy}\mathbf{G} + \mathbf{\Phi}_{xu} = (zI - A)^{-1}B$ and

$$-\mathbf{\Phi}_{uy}\mathbf{G}+\mathbf{\Phi}_{uu}=I.$$

Therefore, the closed-loop responses Φ_{yx} , Φ_{uy} , Φ_{xu} , Φ_{uu} satisfy (27).

Statement 2: Consider any Φ_{xy} , Φ_{uy} , Φ_{xu} , Φ_{uu} satisfying (27). Since $\Phi_{uu} = I + \Phi_{uy}\mathbf{G}$, Φ_{uu} is always invertible. Let $\mathbf{K} = \Phi_{uu}^{-1}\Phi_{uy}$. We now verify that the resulting closed-loop responses (26) are all stable. In particular, we have

$$\mathbf{x} = (zI - A - B\mathbf{K}C)^{-1}B\boldsymbol{\delta}_u,$$

and with the controller $\mathbf{K} = \mathbf{\Phi}_{uu}^{-1} \mathbf{\Phi}_{uu}$, we have

$$(zI - A - B\mathbf{\Phi}_{uu}^{-1}\mathbf{\Phi}_{uy}C)^{-1}B$$

$$= (zI - A)^{-1}(I - B\mathbf{\Phi}_{uu}^{-1}\mathbf{\Phi}_{uy}C(zI - A)^{-1})^{-1}B$$

$$= (zI - A)^{-1}B(I - \mathbf{\Phi}_{uu}^{-1}\mathbf{\Phi}_{uy}\mathbf{G})^{-1}$$

$$= (zI - A)^{-1}B(\mathbf{\Phi}_{uu} - \mathbf{\Phi}_{uy}\mathbf{G})^{-1}\mathbf{\Phi}_{uu}$$

$$= \mathbf{\Phi}_{xu} \in \mathcal{RH}_{\infty}.$$

where the last equality follows from the fact that $\Phi_{xy}, \Phi_{uy}, \Phi_{xu}, \Phi_{uu}$ satisfy (27). Also, it is not difficult to derive that

$$\begin{split} \mathbf{x} &= (zI - A - B\mathbf{K}C)^{-1}B\mathbf{K}\boldsymbol{\delta}_y = \boldsymbol{\Phi}_{xy}\boldsymbol{\delta}_y, \\ \mathbf{u} &= \mathbf{K}(C\boldsymbol{\Phi}_{xx}B\mathbf{K} + I)\boldsymbol{\delta}_y = \boldsymbol{\Phi}_{uy}\boldsymbol{\delta}_y, \\ \mathbf{u} &= (\mathbf{K}C(zI - A - B\mathbf{K}C)^{-1}B + I)\boldsymbol{\delta}_u = \boldsymbol{\Phi}_{uu}\boldsymbol{\delta}_u. \end{split}$$

Thus, we have proved that

$$\left(egin{bmatrix} oldsymbol{\delta}_y \ oldsymbol{\delta}_u \end{bmatrix}
ightarrow egin{bmatrix} \mathbf{x} \ \mathbf{u} \end{bmatrix}
ight) \in \mathcal{RH}_{\infty}.$$

By Theorem 1, we conclude that $\mathbf{K} = \mathbf{\Phi}_{uu}^{-1} \mathbf{\Phi}_{uy} \in \mathcal{C}_{\mathrm{stab}}$.

C Controllability and observability of (28)

Lemma 4 ([1]) The following statements are equivalent: (1) (A, B) is controllable;

- (2) (A + BF, B) is controllable for any compatible matrix F:
- (3) $[A \lambda I, B]$ has full row rank, $\forall \lambda \in \mathbb{C}$.

We are ready to prove the controllability of (A_{cl}, \hat{B}_2) . First, controllability is invariant under state feedback. We consider

$$\begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix} - \begin{bmatrix} BD_k & B \\ B_k & 0 \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix}.$$

Since (A, B) and (A_k, B_k) are controllable, we have $\operatorname{rank}(\lambda I - A, B) = n$ and $\operatorname{rank}(\lambda I - A_k, B_k) = n_k, \forall \lambda \in \mathbb{C}$. Thus,

$$\operatorname{rank}\left(\begin{bmatrix}\lambda I-A & -BC_k & BD_k & B\\ 0 & \lambda I-A_k & B_k & 0\end{bmatrix}\right)=n+n_k, \forall \lambda \in \mathbb{C},$$

indicating that $(A_{\rm cl}, \hat{B}_2)$ is controllable. The observability of $(A_{\rm cl}, \hat{C}_1)$ can be proved in a similar way.

D Proofs of Theorems 3 and 4

In this section, we use the following system operations very often (see [1, Chapter 3.6]). Consider two dynamic systems

$$\mathbf{G}_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad i = 1, 2.$$

Their inverses are given by

$$\mathbf{G}_{i}^{-1} = \begin{bmatrix} A_{i} - B_{i}D_{i}^{-1}C_{i} & -B_{i}D_{i}^{-1} \\ D_{i}^{-1}C_{i} & D_{i}^{-1} \end{bmatrix},$$

where we assume D_i is invertible. If the system is strictly proper, then the inverse will be non-proper and there is no state-space realization. The cascade connection of two systems such that $\mathbf{y} = \mathbf{G_1}\mathbf{G_2}\mathbf{u}$ has a realization

$$\mathbf{G}_{1}\mathbf{G}_{2} = \begin{bmatrix} A_{1} & B_{1}C_{2} & B_{1}D_{2} \\ 0 & A_{2} & B_{2} \\ \hline C_{1} & D_{1}C_{2} & D_{1}D_{2} \end{bmatrix}$$
(D.1)

and a parallel connection $\mathbf{y} = (\mathbf{G}_1 - \mathbf{G}_2)\mathbf{u}$ has a realization

$$\mathbf{G}_{1} - \mathbf{G}_{2} = \begin{bmatrix} A_{1} & 0 & B_{1} \\ 0 & A_{2} & B_{2} \\ \hline C_{1} & -C_{2} & D_{1} - D_{2} \end{bmatrix}.$$
 (D.2)

We note that (D.1) and (D.2) are in general not minimal. In addition, we use the following fact: for any invertible

matrix T with proper dimension,

$$\mathbf{G}_{i} = \begin{bmatrix} A_{i} & B_{i} \\ C_{i} & D_{i} \end{bmatrix} = \begin{bmatrix} TA_{i}T^{-1} & TB_{i} \\ C_{i}T^{-1} & D_{i} \end{bmatrix}. \tag{D.3}$$

Proof of Theorem 3

Considering the FIR transfer matrices Φ_{uy} and Φ_{yy} in (30). By the affine constraint (23), we always have $Y_0 = I_p$. The following state-space realizations of Φ_{uy} and Φ_{uy} are

$$oldsymbol{\Phi}_{uy} = egin{bmatrix} Z_p & \mathcal{I}_p \ \hat{U} & U_0 \end{bmatrix}, \quad oldsymbol{\Phi}_{yy} = egin{bmatrix} Z_p & \mathcal{I}_p \ \hat{Y} & I_p \end{bmatrix},$$

with \hat{U} and \hat{Y} defined in (30), and Z_p and \mathcal{I}_p defined as

$$Z_p = egin{bmatrix} 0 & 0 & 0 & \dots & 0 \ I_p & 0 & 0 & \dots & 0 \ 0 & I_p & 0 & \dots & 0 \ dots & dots & \ddots & \ddots & dots \ 0 & 0 & \dots & I_p & 0 \ \end{pmatrix} \in \mathbb{R}^{pT imes pT}, \mathcal{I}_p = egin{bmatrix} I_p \ 0 \ dots \ 0 \ \end{bmatrix} \in \mathbb{R}^{pT imes p}.$$

Then, we have

$$\begin{split} \boldsymbol{\Phi}_{uy} \boldsymbol{\Phi}_{yy}^{-1} &= \left[\frac{Z_p \left| \mathcal{I}_p \right|}{\hat{U} \left| U_0 \right|} \right] \left[\frac{Z_p \left| \mathcal{I}_p \right|}{\hat{Y} \left| I_p \right|} \right]^{-1} \\ &= \left[\frac{Z_p \left| \mathcal{I}_p \right|}{\hat{U} \left| U_0 \right|} \right] \left[\frac{Z_p - \mathcal{I}_p \hat{Y} \left| -\mathcal{I}_p \right|}{\hat{Y} \left| I_p \right|} \right] \\ &= \left[\frac{Z_p \left| \mathcal{I}_p \right|}{\hat{U} \left| U_0 \right|} \right] \left[\frac{Z_p - \mathcal{I}_p \hat{Y} \left| \mathcal{I}_p \right|}{\hat{Y} \left| U_0 \right|} \right]. \end{split}$$

By defining a transformation

$$T = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}, T^{-1} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix},$$

with compatible dimension, and according to (D.3), we have

$$\mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1} = \begin{bmatrix} Z_p & 0 & 0\\ 0 & Z_p - \mathcal{I}_p \hat{Y} & -\mathcal{I}_p\\ \hat{U} & U_0 \hat{Y} - \hat{U} & U_0 \end{bmatrix} = \begin{bmatrix} Z_p - \mathcal{I}_p \hat{Y} & -\mathcal{I}_p\\ U_0 \hat{Y} - \hat{U} & U_0 \end{bmatrix}.$$

In the last step, we have removed some uncontrollable and unobservable modes.

Proof of Theorem 4

First, similar to the realization of $\Phi_{uy}\Phi_{yy}^{-1}$, we have

$$\Phi_{ux}\Phi_{xx}^{-1} = (z\Phi_{ux})(z\Phi_{xx})^{-1} = \left[\frac{Z_n - \mathcal{I}_n \hat{R} - \mathcal{I}_n}{M_1 \hat{R} - \hat{M} M_1} \right],$$

where \hat{M} and \hat{R} are defined in (34), and Z_n and \mathcal{I}_n are defined as

$$Z_{n} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ I_{n} & 0 & 0 & \dots & 0 \\ 0 & I_{n} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & I_{n} & 0 \end{bmatrix} \in \mathbb{R}^{n\hat{T} \times n\hat{T}}, \mathcal{I}_{n} = \begin{bmatrix} I_{n} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n\hat{T} \times n}.$$

Then, we have

$$\begin{aligned} & \Phi_{uy} - (z\Phi_{ux})(z\Phi_{xx})^{-1}\Phi_{xy} \\ & = \left[\frac{Z_p | \mathcal{I}_p}{\hat{L} | L_0} \right] - \left[\frac{Z_n - \mathcal{I}_n \hat{R} - \mathcal{I}_n \hat{N} | 0}{0 & Z_p | \mathcal{I}_p} \right] \\ & = \left[\frac{Z_p | \mathcal{I}_p}{\hat{L} | L_0} \right] - \left[\frac{Z_n - \hat{\mathcal{I}}_n \hat{R} - \hat{M} | M_1 \hat{N} | 0}{0 | M_1 \hat{R} - \hat{M} | M_1 \hat{N} | 0} \right] \\ & = \left[\frac{Z_p | 0 | 0 | | \mathcal{I}_p}{0 | Z_n - \mathcal{I}_n \hat{R} - \mathcal{I}_n \hat{N} | 0} \right] \\ & = \left[\frac{\partial_{x_1} | \mathcal{I}_p}{\partial_{x_1} | \mathcal{I}_p} - \mathcal{I}_n \hat{R} - \mathcal{I}_n \hat{N} | \mathcal{I}_p} \right] \end{aligned}$$

Define a similarity transformation

$$\hat{T} = \begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \hat{T}^{-1} = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

We derive

$$\mathbf{\Phi}_{uy} - \mathbf{\Phi}_{ux}\mathbf{\Phi}_{xx}^{-1}\mathbf{\Phi}_{xy} = \begin{bmatrix} Z_n - \mathcal{I}_n\hat{R} & -\mathcal{I}_n\hat{N} & 0\\ 0 & Z_p & \mathcal{I}_p\\ \hline -M_1\hat{R} + \hat{M} & -M_1\hat{N} + \hat{L} & L_0 \end{bmatrix}.$$

E Derivations of (40)

For notational convenience, we use $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ in place of $(\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy})$ here, as used in [6]. The following derivations utilize the affine relationship (38) multiple times:

$$(zI - A - B\mathbf{K}C)^{-1}$$

$$= (zI - A - B(\mathbf{L} - \mathbf{M}\mathbf{M}^{-1}\mathbf{N})C)^{-1}$$

$$= (zI - A - B\mathbf{L}C + B\mathbf{M}\mathbf{R}^{-1}\mathbf{N}C)^{-1}$$

$$= (zI - A - B(\mathbf{M}(zI - A) - \hat{\Delta}_4) + B\mathbf{M}\mathbf{R}^{-1}\mathbf{N}C)^{-1}$$

$$= (zI - A - B\mathbf{M}(zI - A) + B\hat{\Delta}_4 + B\mathbf{M}\mathbf{R}^{-1}\mathbf{N}C)^{-1}$$

$$= (zI - A - B\mathbf{M}\mathbf{R}^{-1}(\mathbf{R}(zI - A) - \mathbf{N}C) + B\hat{\boldsymbol{\Delta}}_{4})^{-1}$$

$$= (zI - A - B\mathbf{M}\mathbf{R}^{-1}(I + \hat{\boldsymbol{\Delta}}_{3}) + B\hat{\boldsymbol{\Delta}}_{4})^{-1}$$

$$= (zI - A - B\mathbf{M}\mathbf{R}^{-1} - B\mathbf{M}\mathbf{R}^{-1}\hat{\boldsymbol{\Delta}}_{3} + B\hat{\boldsymbol{\Delta}}_{4})^{-1}$$

$$= \mathbf{R}((zI - A)\mathbf{R} - B\mathbf{M} - B\mathbf{M}\mathbf{R}^{-1}\hat{\boldsymbol{\Delta}}_{3}\mathbf{R} + B\hat{\boldsymbol{\Delta}}_{4}\mathbf{R})^{-1}$$

$$= \mathbf{R}(I + \hat{\boldsymbol{\Delta}}_{1} - B\mathbf{M}\mathbf{R}^{-1}\hat{\boldsymbol{\Delta}}_{3}\mathbf{R} + B\hat{\boldsymbol{\Delta}}_{4}\mathbf{R})^{-1}$$

$$= \mathbf{R}(I + \hat{\boldsymbol{\Delta}}_{1} - ((zI - A)\mathbf{R} - I - \hat{\boldsymbol{\Delta}}_{1})\mathbf{R}^{-1}\hat{\boldsymbol{\Delta}}_{3}\mathbf{R} + B\hat{\boldsymbol{\Delta}}_{4}\mathbf{R})^{-1}.$$
We can further simplified this expression:

$$\mathbf{R}(I + \hat{\boldsymbol{\Delta}}_1 - ((zI - A)\mathbf{R} - I - \hat{\boldsymbol{\Delta}}_1)\mathbf{R}^{-1}\hat{\boldsymbol{\Delta}}_3\mathbf{R} + B\hat{\boldsymbol{\Delta}}_4\mathbf{R})^{-1}$$

$$= ((I + \hat{\boldsymbol{\Delta}}_1)\mathbf{R}^{-1} - (zI - A)\hat{\boldsymbol{\Delta}}_3 + (I + \hat{\boldsymbol{\Delta}}_1)\mathbf{R}^{-1}\hat{\boldsymbol{\Delta}}_3 + B\hat{\boldsymbol{\Delta}}_4)^{-1}$$

$$= (I + \hat{\boldsymbol{\Delta}})^{-1}\mathbf{R}(I + \hat{\boldsymbol{\Delta}}_1)^{-1},$$

where $\hat{\Delta}$ is defined in (39).

F Robustness of Mixed I/II parameterizations

We consider the Mixed I parameterization in Proposition 3. The transfer matrices $\hat{\Phi}_{yx}$, $\hat{\Phi}_{ux}$, $\hat{\Phi}_{yy}$, $\hat{\Phi}_{uy}$ only approximately satisfy the affine constraint (25), *i.e.*, we have

$$\begin{bmatrix} I - \mathbf{G} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{yx} & \hat{\mathbf{\Phi}}_{yy} \\ \hat{\mathbf{\Phi}}_{ux} & \hat{\mathbf{\Phi}}_{uy} \end{bmatrix} = \begin{bmatrix} C(zI - A)^{-1} + \mathbf{\Delta}_1 & I + \mathbf{\Delta}_2 \end{bmatrix},$$

$$\begin{bmatrix} \hat{\mathbf{\Phi}}_{yx} & \hat{\mathbf{\Phi}}_{yy} \\ \hat{\mathbf{\Phi}}_{ux} & \hat{\mathbf{\Phi}}_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_3 \\ \mathbf{\Delta}_4 \end{bmatrix},$$

$$\hat{\mathbf{\Phi}}_{yx}, \hat{\mathbf{\Phi}}_{ux}, \hat{\mathbf{\Phi}}_{yy}, \hat{\mathbf{\Phi}}_{uy} \in \mathcal{RH}_{\infty},$$
(F.1)

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the computational residuals. **Theorem 7** Let $\hat{\Phi}_{yx}, \hat{\Phi}_{ux}, \hat{\Phi}_{yy}, \hat{\Phi}_{uy}$ satisfy (F.1). Then, we have the following statements.

- (1) In the case of $\mathbf{G} \in \mathcal{RH}_{\infty}$, the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uy}\hat{\mathbf{\Phi}}_{yy}^{-1}$ internally stabilizes the plant \mathbf{G} if and only if $(I + \mathbf{\Delta}_2)^{-1}$ is stable.
- (2) In the case of G ∉ RH_∞, the controller K = Φ̂_{uy}Φ̂⁻¹_{yy} cannot guarantee the internal stability of the closed-loop system unless Δ₁ = 0, Δ₂ = 0, Δ₃ = 0, Δ₄ = 0.
 The proof is almost identical to that of Theorem 5.

The proof is almost identical to that of Theorem 5.

We then consider Mixed II parameterization in Proposition 4. The transfer matrices $\hat{\Phi}_{xy}$, $\hat{\Phi}_{uy}$, $\hat{\Phi}_{xu}$, $\hat{\Phi}_{uu}$ only approximately satisfy the affine constraint (27), *i.e.*, we have

$$\begin{bmatrix} zI - A - B \end{bmatrix} \begin{bmatrix} \hat{\mathbf{\Phi}}_{xy} & \hat{\mathbf{\Phi}}_{xu} \\ \hat{\mathbf{\Phi}}_{uy} & \hat{\mathbf{\Phi}}_{uu} \end{bmatrix} = \begin{bmatrix} \mathbf{\Delta}_{1} & \mathbf{\Delta}_{2} \end{bmatrix},$$

$$\begin{bmatrix} \hat{\mathbf{\Phi}}_{xy} & \hat{\mathbf{\Phi}}_{xu} \\ \hat{\mathbf{\Phi}}_{uy} & \hat{\mathbf{\Phi}}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = \begin{bmatrix} (zI - A)^{-1}B + \mathbf{\Delta}_{3} \\ I + \mathbf{\Delta}_{4} \end{bmatrix},$$

$$\hat{\mathbf{\Phi}}_{xy}, \hat{\mathbf{\Phi}}_{uy}, \hat{\mathbf{\Phi}}_{xu}, \hat{\mathbf{\Phi}}_{uu} \in \mathcal{RH}_{\infty},$$
(F.2)

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are the computational residuals.

Theorem 8 Let $\hat{\Phi}_{xy}$, $\hat{\Phi}_{uy}$, $\hat{\Phi}_{xu}$, $\hat{\Phi}_{uu}$ satisfy (F.2). Then, we have the following statements.

(1) In the case of $\mathbf{G} \in \mathcal{RH}_{\infty}$, the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uu}^{-1}\hat{\mathbf{\Phi}}_{uy}$ internally stabilizes the plant \mathbf{G} if and only if $(I + \mathbf{\Delta}_4)^{-1}$ is stable.

(2) In the case of $\mathbf{G} \notin \mathcal{RH}_{\infty}$, the controller $\mathbf{K} = \hat{\mathbf{\Phi}}_{uu}^{-1}\hat{\mathbf{\Phi}}_{uy}$ cannot guarantee the internal stability of the closed-loop system unless $\mathbf{\Delta}_1 = 0, \mathbf{\Delta}_2 = 0, \mathbf{\Delta}_3 = 0, \mathbf{\Delta}_4 = 0$.

The proof is similar to that of Theorem 5. We just highlight that for the case $\mathbf{G} \in \mathcal{RH}_{\infty}$, we only need to check the closed-loop response from $\boldsymbol{\delta}_y$ to \boldsymbol{u} , which is

$$u = \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\boldsymbol{\delta}_{y}$$

$$= (I - \mathbf{K}\mathbf{G})^{-1}\mathbf{K}\boldsymbol{\delta}_{y}$$

$$= (I - \hat{\mathbf{\Phi}}_{uu}^{-1}\hat{\mathbf{\Phi}}_{uy}\mathbf{G})^{-1}\hat{\mathbf{\Phi}}_{uu}^{-1}\hat{\mathbf{\Phi}}_{uy}\boldsymbol{\delta}_{y}$$

$$= (I + \mathbf{\Delta}_{4})^{-1}\hat{\mathbf{\Phi}}_{uy}\boldsymbol{\delta}_{y}.$$

Then, the first statement in Theorem 8 becomes obvious.

G Proof of Proposition 5

Proof: \Leftarrow We first prove that $\forall \mathbf{K} \in \hat{\mathcal{C}}_{stab}$, we have $\mathbf{K} \in \mathcal{C}_{stab}$. Since $\mathbf{K} \in \hat{\mathcal{C}}_{stab}$, we know

$$\begin{bmatrix} (I - \hat{\mathbf{G}} \mathbf{K}_1)^{-1} & (I - \hat{\mathbf{G}} \mathbf{K}_1)^{-1} \hat{\mathbf{G}} \\ \mathbf{K}_1 (I - \hat{\mathbf{G}} \mathbf{K}_1)^{-1} & (I - \mathbf{K}_1 \hat{\mathbf{G}})^{-1} \end{bmatrix} \in \mathcal{RH}_{\infty},$$

where $\hat{\mathbf{G}} = (I - \mathbf{G}\mathbf{K}_0)^{-1}\mathbf{G}$ and $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_0$ with $\mathbf{K}_0 \in \mathcal{C}_{\text{stab}} \cap \mathcal{RH}_{\infty}$. Now, we verify that

$$(I - \mathbf{G}(\mathbf{K}_0 + \mathbf{K}_1))^{-1} = (I - \hat{\mathbf{G}}\mathbf{K}_1)^{-1}(I - \mathbf{G}\mathbf{K}_0)^{-1} \in \mathcal{RH}_{\infty}$$
$$(I - \mathbf{G}(\mathbf{K}_0 + \mathbf{K}_1))^{-1}\mathbf{G} = (I - \hat{\mathbf{G}}\mathbf{K}_1)^{-1}\hat{\mathbf{G}} \in \mathcal{RH}_{\infty}$$

and that

$$\begin{aligned} & (\mathbf{K}_0 + \mathbf{K}_1)(I - \mathbf{G}\mathbf{K})^{-1} \\ = & \mathbf{K}_0(I - \mathbf{G}\mathbf{K})^{-1} + \mathbf{K}_1(I - \mathbf{G}\mathbf{K})^{-1} \in \mathcal{RH}_{\infty}. \end{aligned}$$

Finally, we show

$$(I - (\mathbf{K}_0 + \mathbf{K}_1)\mathbf{G})^{-1} = (I - \mathbf{K}_1\hat{\mathbf{G}})^{-1}(I - \mathbf{K}_0\mathbf{G})^{-1} \in \mathcal{RH}_{\infty}.$$

By Theorem 1, we have proved $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1 \in \mathcal{C}_{\mathrm{stab}}$.

 \Rightarrow : This direction is similar: $\forall \mathbf{K} \in \mathcal{C}_{\mathrm{stab}}$, we prove that $\mathbf{K}_1 = \mathbf{K} - \mathbf{K}_0$ internally stabilizes $\hat{\mathbf{G}}$. For example

$$(I - \hat{\mathbf{G}}\mathbf{K}_1)^{-1} = (I - \mathbf{G}\mathbf{K})^{-1}(I + \mathbf{G}\mathbf{K}_0)$$

= $(I - \mathbf{G}\mathbf{K})^{-1} + (I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G}\mathbf{K}_0 \in \mathcal{RH}_{\infty}$

and other conditions can be proved similarly. We complete the proof. $\hfill\Box$

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