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A Chordal Decomposition Approach to Scalable Design of Structured Feedback Gains over Directed Graphs

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Abstract— **This paper considers the problem of designing static feedback gains subject to *a priori* structural constraints, which is in general a non-convex problem. By exploiting the sparsity properties of the problem, and using chordal decomposition, a scalable algorithm is proposed to compute structured stabilizing feedback gains for large-scale systems over directed graphs. Specifically, we first present a chordal decomposition theorem for block-semidefinite matrices. A relaxation is then used to recast the design of structured feedback gains into a convex problem. Combining the decomposition with the relaxation, we propose a sequential design algorithm to obtain structured feedback gains clique-by-clique over a clique tree of the underlying chordal graph. Numerical simulations demonstrate the efficiency of the proposed method.**

I. Introduction

Controller synthesis for interconnected decentralized systems has recently received considerable attention [1]–[4]. This problem arises in a wide range of engineering applications, such as the smart grid [5] and automated highways [6]. One key challenge in decentralized systems is to design structured control policies based on local information, aiming to stabilize a large-scale system and further minimize a certain quadratic performance measure, such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$. In fact, it has been shown that the general problem of designing feedback gains subject to structured constraints is NP-hard [7]. Previous approaches to synthesize decentralized controllers with information constraints can be roughly categorized into three groups: 1) finding exact solutions for special structures, *e.g.*, *quadratically invariant* [4] and partially ordered sets [8]; 2) seeking tractable design approaches via convex approximations [9], [10]; and 3) obtaining suboptimal solutions by solving the non-convex problem directly, using, *e.g.*, augmented Lagrangian [11] and alternating direction method of multipliers (ADMM) approaches [12].

Despite the aforementioned results that provide powerful tools for controller synthesis of decentralized systems, there is relatively less focus on the algorithmic aspects that could make these methods practical and scalable for realistic large-scale systems. As a result, most of the illustration examples in the literature are small-scale systems. However, some practical systems, such as the power grid and transportation systems, could contain thousands of states and controls. The objective of this paper is to develop an efficient algorithm to design static structured feedback gains for large-scale systems, by utilizing the properties of positive semidefinite matrices and chordal graphs.

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Chordal graphs are very well studied objects in graph theory [13]. Several important problems that are hard on general graphs can be solved in polynomial time in the case of chordal graphs, such as the graph colouring problem [14]. Grone *et al.* [15] and Agler *et al.* [16] introduced two important results that connect positive sparse semidefinite matrices and chordal graphs. Furthermore, Fukuda *et al.* [17] and Kim *et al.* [18] showed that the results in [15], [16] can be used to decompose the conic constraint in primal and dual SDPs, respectively. Recently, these results have been applied to the stability analysis of large-scale linear systems in [19], [20], obtaining faster solutions than using standard methods.

In this paper, we introduce a scalable sequential design algorithm to synthesize structured feedback gains for large-scale systems using chordal decomposition. We use two directed graphs to model the interconnected system: a plant graph and a communication graph. This naturally leads to block structured constraints in controller synthesis. The design of structured controllers is first relaxed into a convex problem using a block-diagonal matrix assumption, which leads to a decomposition of coupled subsystems over the maximal cliques of the underlying graph. Then, a sequential algorithm is proposed to compute structured feedback gains clique-by-clique over a clique tree. Illustrative examples are used to show the efficiency of the proposed method.

II. Preliminaries and Problem Statement

We begin this section with a brief introduction on chordal graphs (see [13] for more details). The last part of this section presents the problem statement.

A. Chordal Graphs

A directed graph $\mathcal{G}$ is denoted by a set of vertices $\mathcal{V} = \{1, 2 \ldots, N\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote $(i, j) \in \mathcal{E}$ if there is a directed edge from vertex i to vertex j. We assume that graph $\mathcal{G}$ has no self-loops, *i.e.*, $(i, i) \notin \mathcal{E}$. For each vertex $i \in \mathcal{V}$, the set of its neighbours is defined as $\mathbb{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. A cycle of length k is a sequence of pairwise distinct vertices $(v_1, v_2, \ldots, v_k)$ such that $(v_k, v_1) \in \mathcal{E}$ and $(v_i, v_{i+1}) \in \mathcal{E}$ for $i = 1, \ldots, k-1$. A chord is an edge joining two non-adjacent vertices in a cycle. A graph $\mathcal{G}$ is *undirected* if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$.

Definition 1: An undirected graph is chordal if every cycle of length greater than or equal to 4 has a chord.

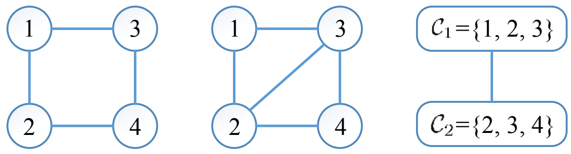

Fig. 1: Example of chordal extension and clique tree: (a) nonchordal graph, (b) chordal graph, (c) clique tree.

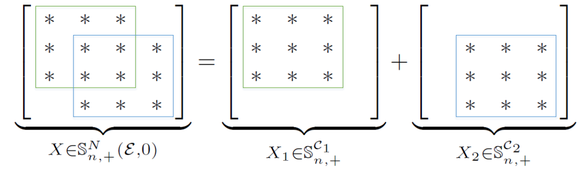

Fig. 2: Illustration of Theorem 1 for the graph in Fig. 1(b). $X \in \mathbb{S}^N_{n,+}(\mathcal{E},0)$ can be decomposed as a sum of X_i , where $X_i \in \mathbb{S}^{\mathcal{C}_i}_{n,+}, i = 1, 2$.

A clique of graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subset of vertices $\mathcal{C} \subseteq \mathcal{V}$ such that $(i, j) \in \mathcal{E}$ for any distinct vertices $i, j \in \mathcal{C}$. The clique is called maximal if it is not a subset of another clique. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a chordal graph with a set of maximal cliques $\Gamma = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. These cliques can be further rearranged in a clique tree $\mathcal{T} = (\Gamma, \Xi)$ with $\Xi \subseteq \Gamma \times \Gamma$, which satisfies the *running intersection property*, *i.e.*, $\mathcal{C}_i \cap \mathcal{C}_j \subseteq \mathcal{C}_k$ if clique $\mathcal{C}_k$ lies on the path between cliques $\mathcal{C}_i$ and $\mathcal{C}_j$ in the tree [13]. Note that some maximal cliques have overlapping vertices. Let $\mathcal{C}$ be an arbitrary subset of $\mathcal{V}$ and define a set $J(\mathcal{C}) = \{(i, j) \in \mathcal{C} \times \mathcal{C} \mid i \le j\}$. Then, given a clique tree $\mathcal{T} = (\Gamma, \Xi)$, we denote the minimal set of overlapping elements by $\Lambda = \{(i, j, k, l) \mid (i, j) \in J(\mathcal{C}_k \cap \mathcal{C}_l), (\mathcal{C}_k, \mathcal{C}_l) \in \Xi\}$.

Nonchordal graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be *chordal extended*, *i.e.*, we can add additional edges to $\mathcal{E}$ to construct a chordal graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}')$. Several heuristics, such as the minimum degree ordering followed by a symbolic Cholesky factorization, are known to construct a good chordal extension efficiently [13]. Fig. 1 illustrates some of these notions.

B. Sparsity Structures and Chordal Decomposition

Given a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with no self-loops, we define $\hat{\mathcal{E}} = \mathcal{E} \cup \{(i, i), i = 1, \ldots, N\}$. The set of matrices with a sparsity structure characterized by $\mathcal{G}$ is defined as:

$$\mathbb{R}^N_{m,n}(\mathcal{E}, 0) = \{X \in \mathbb{R}^{mN \times nN} \mid X_{ij} = 0 \text{ if } (j, i) \notin \hat{\mathcal{E}}\},$$

where X_{ij} is a block of size $m \times n$. When each block is square, we simplify $\mathbb{R}^N_{n,n}(\mathcal{E}, 0)$ to $\mathbb{R}^N_n(\mathcal{E}, 0)$. If $\mathcal{G}$ is undirected, we further define the following sets:

$$\begin{aligned}
&\mathbb{S}^N_n(\mathcal{E}, 0) = \{X \in \mathbb{S}^{nN} \mid X_{ij} = 0 \text{ if } (j, i) \notin \hat{\mathcal{E}}\},\\
&\mathbb{S}^N_{n,+}(\mathcal{E}, 0) = \{X \in \mathbb{S}^N_n(\mathcal{E}, 0) \mid X \succeq 0\},\\
&\mathbb{S}^{\mathcal{C}}_n = \{X \in \mathbb{S}^{nN} \mid X_{ij} = 0 \text{ if } (i, j) \notin \mathcal{C} \times \mathcal{C}\} \text{ for } \mathcal{C} \subseteq \mathcal{V},\\
&\mathbb{S}^{\mathcal{C}}_{n,+} = \{X \in \mathbb{S}^{\mathcal{C}}_n \mid X \succeq 0\}.
\end{aligned}$$

It is also convenient to define (block) submatrices based on the subsets of $\mathcal{V}$. Given a block matrix $X \in \mathbb{R}^{mN \times nN}$ and two subsets $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{V}$, we define

$$X(\mathcal{C}_1, \mathcal{C}_2) = \Big\{\hat{X} \in \mathbb{R}^{mN \times nN} \mid \hat{X}_{ij} = X_{ij} \text{ if } (i, j) \in \mathcal{C}_1 \times \mathcal{C}_2, \text{ otherwise, } \hat{X}_{ij} = 0\Big\}.$$

Here, we introduce the first result of this paper.

Theorem 1: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a chordal graph with a set of maximal cliques $\Gamma = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. Then, $X = [X_{ij}]_{N \times N} \in \mathbb{S}^N_n(\mathcal{E}, 0)$ is positive semidefinite if and only if there exists a set of matrices X_k, which decomposes X as $X = \sum_{k=1}^p X_k$, where $X_k \in \mathbb{S}^{\mathcal{C}_k}_{n,+}, k = 1, \ldots, p$.

The proof is omitted here for brevity, and will be reported elsewhere. This result is referred to as the chordal decomposition theorem in this paper. Note that Theorem 1 does not impose any restrictions on the size of each block, *i.e.*, n can be any integer. When $n = 1$, Theorem 1 is reduced to Agler's theorem [15]. Fig. 2 gives an illustration of Theorem 1 for the chordal graph shown in Fig. 1(b).

Note also that this theorem presents an attractive connection between chordal graphs and block positive semidefinite matrices, which will be used to improve the computational efficiency of structured feedback gains for large-scale systems in Section IV.

C. Problem Statement: Large-scale Systems over Graphs

This paper considers decentralized systems over directed graphs with a vertex set $\mathcal{V}$, in which each vertex represents a subsystem and a corresponding controller. In reality, a large-scale system consists of two underlying graphs: 1) a plant graph $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$, characterizing the dynamic coupling of the plant; and 2) a communication graph $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c)$, indicating the allowable communication of the controllers. See the example of hierarchical systems shown in Fig. 3.

$\mathcal{G}^p$ and $\mathcal{G}^c$ are in general different directed graphs. Some previous work focused on special graph structures. For instance, Shah and Parrilo assumed these graphs could be modelled by partial order sets [8]. For dynamically decoupled plants, such as in the platoon control problem [21], $\mathcal{G}^p$ has no edges. Also, $\mathcal{G}^c$ would have no edges if there exists no communication between subsystems (referred to as fully decentralized systems). The existence of a stabilizing controller for fully decentralized systems was investigated in [22], and a generalized version was recently reported in [23]. In this paper, we do not restrict $\mathcal{G}^p$ or $\mathcal{G}^c$.

For each subsystem i, the state $x_i(t) \in \mathbb{R}^n$ evolves as

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j \in \mathbb{N}^p_i} A_{ij}x_j(t) + B_i u_i(t),$$

where $u_i(t) \in \mathbb{R}^m$ is the control input, and $\mathbb{N}^p_i$ denotes the vertices that exert influence on the dynamics of vertex i in $\mathcal{G}^p$. The overall state-space system is then given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) = [x_1(t)^T, \ldots, x_N(t)^T]^T$ and similarly for $u(t)$. Note that $A \in \mathbb{R}^N_n(\mathcal{E}^p, 0)$, and $B = \text{diag}\{B_1, \ldots, B_N\}$.

Our objective is to stabilize (1) by designing the control input $u(t)$ based on the local information defined by $\mathcal{G}^c$. In

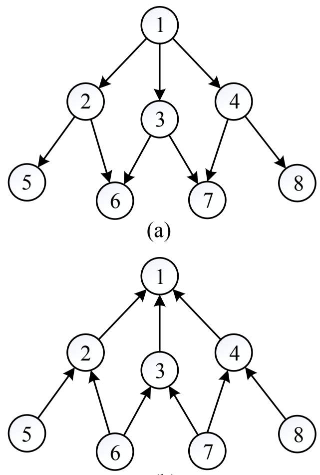

Fig. 3: Example of hierarchical systems. (a) Graph $\mathcal{G}^p = (\mathcal{V}, \mathcal{E}^p)$; here, only dynamics of subsystems in the upper layer have influence on those in the lower layer. (b) Graph $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c)$; here, only the nodes in the upper layer can use the state information of nodes in the lower layer.

this paper, static state feedback is used, as in [12]. Additionally, we assume communication conditions are perfect in terms of no time-delays or bandwidth restrictions. We are looking for controllers of the form

$$u_i(t) = k_{ii}x_i(t) + \sum_{j \in \mathbb{N}_i^c} k_{ij}x_j(t), \tag{2}$$

where $\mathbb{N}_i^c$ denotes the vertices that send their state information to vertex i in graph $\mathcal{G}^c$. The compact form of the overall controller is denoted by

$$u(t) = Kx(t),\ K \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0), \tag{3}$$

and the closed-loop system is

$$\begin{aligned} \dot{x}(t) &= (A + BK)x(t), \\ A \in \mathbb{R}^N_n(\mathcal{E}^p, 0)&, K \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0). \end{aligned} \tag{4}$$

Concisely, the problem considered in this paper is as follows

$$\begin{aligned} &\text{Find } K \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0), \\ &\text{such that } A + BK \text{ is asymptotically stable.} \end{aligned} \tag{5}$$

There exist many well-known methods to compute the controller in (5) if there are no structural constraints on K. However, sparsity constraints arise naturally in the design of decentralized systems. In general, such seemingly mild and natural requirements actually make the problem challenging [7]. Previous work either imposed special structures or used certain relaxation techniques to solve this problem, as well as to minimize a certain performance measure (typically $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm) [4], [8]–[11]. On the other hand, the sparsity in matrices A, K has the potential to bring certain benefits from the perspective of numerical computations. The speed and accuracy of numerically computing a controller can actually be improved if this sparsity is taken advantage of.

In this paper, we focus on the structured stabilization problem (5), and propose a scalable algorithm for large-scale decentralized systems by exploiting properties of chordal graphs and sparse positive semidefinite matrices.

III. Design of Structured Feedback Gains using Convex Relaxation

In this section, a relaxation technique is introduced to convert problem (5) into a linear matrix inequality (LMI) that inherits the problem's sparsity pattern. The scalable design algorithm that uses chordal decomposition will be developed in the next section.

Recall that conditions for stability can be equivalently expressed as the following inequalities:

$$\begin{cases} QA^T + AQ + R^TB^T + BR \prec 0 \\ RQ^{-1} \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0) \\ Q \succ 0 \end{cases}. \tag{6}$$

The steps to obtain condition (6) are well known, and involve the use of a Lyapunov function $V(x) = x^TPx$, where P is a positive definite matrix of compatible dimensions; $Q = P^{-1}$, and $R = KQ$.

The structural constraint of communication graph $\mathcal{G}^c$, which is nonlinear, can be relaxed if we assume that Q (and hence Q^{-1}) is block diagonal with block sizes compatible to those of the subsystems, which leads to:

$$RQ^{-1} \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0) \Leftrightarrow R \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0). \tag{7}$$

This assumption convexifies the problem (6) into

$$\begin{cases} QA^T + AQ + R^TB^T + BR \prec 0 \\ R \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0) \\ Q \succ 0, Q \text{ is block diagonal} \end{cases}, \tag{8}$$

but this is still centralized.

The assumption that the closed-loop system admits a block diagonal Lyapunov function will introduce conservativeness for general large-scale systems. However, many large-scale systems, such as transportation networks and power systems, are positive systems, whose stability is equivalent to the existence of diagonal Lyapunov functions [24]. Moreover, it is has been shown that this introduces no conservativeness for computing the $\mathcal{H}_\infty$ norm of positive systems in [25]. Besides, if the system's dynamics are in block lower triangular forms, *e.g.*, systems modeled by a poset, stability is also equivalent to the existence of a block diagonal Lyapunov function [8]. Therefore, the relaxation technique (7) is practical and acceptable. More importantly, the resulting convex problem (8) inherits the sparsity pattern of (5), which allows the use of chordal decomposition in the next section.

Problem (8) is ready to be solved to obtain structured feedback gains via convex optimization in a centralized way. However, both the computational efficiency and quality of the solution will become worse as the systems become larger, since the size of resulting LMI scales as nN. In the next section, we turn to establish a scalable algorithm based on Theorem 1 to solve (8), when system (4) is large and sparse.

IV. The Scalable Solution via Chordal Decomposition

This section proposes a scalable sequential method to obtain structured feedback gains by applying Theorem 1 to the convex relaxation (8). We first present a method to get a chordal characterization of the system data in (8), which directly leads to decomposition of the positive semidefinite constraints by applying Theorem 1. Then, a sequential method is derived by *a priori* dividing the overlapping elements equally in the decomposed subsystems.

A. Chordal Characterization of System Data

The matrices A, K in original problem (5) have sparsity patterns characterized by the directed graphs $\mathcal{G}^p$ and $\mathcal{G}^c$, respectively. However, the sparsity pattern of the Lyapunov condition (8) corresponds to an undirected super-graph covering both $\mathcal{G}^p$ and $\mathcal{G}^c$. Considering the assumption of block diagonal Q, we know

$$AQ \in \mathbb{R}_n^N(\mathcal{E}^p, 0), BR \in \mathbb{R}_n^N(\mathcal{E}^c, 0). \tag{9}$$

To handle the symmetry in (8), we introduce the notion of mirror graphs.

Definition 2: (*Mirror Graph*) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph. We define $\mathcal{E}_m$ as a set of reverse edges of $\mathcal{G}$ obtained by reversing the order of nodes in all the pairs in $\mathcal{E}$. The mirror of $\mathcal{G}$ denoted by $\mathcal{G}_m = \mathcal{M}(\mathcal{G})$ is a directed graph in the form $\mathcal{G}_m = (\mathcal{V}, \mathcal{E}_m)$ with the same set of nodes $\mathcal{V}$ and the set of reverse edges $\mathcal{E}_m$.

As an example, it is easy to know that the graphs in Fig. 3 (a) and (b) are mirror graphs of each other. Then, we have

$$QA^T \in \mathbb{R}_n^N(\mathcal{E}_m^p, 0), R^T B^T \in \mathbb{R}_n^N(\mathcal{E}_m^c, 0), \tag{10}$$

where $\mathcal{G}_m^p = (\mathcal{V}, \mathcal{E}_m^p), \mathcal{G}_m^c = (\mathcal{V}, \mathcal{E}_m^c)$ are the mirror graphs of $\mathcal{G}_m^p$, $\mathcal{G}_m^c$, respectively.

We further define an undirected super-graph $\mathcal{G}_s = (\mathcal{V}, \mathcal{E}_s)$ to include both the dynamical coupling of plants $\mathcal{G}^p$ and communication connections of controllers $\mathcal{G}^c$:

$$\mathcal{G}_s = \mathcal{G}^p \cup \mathcal{G}_m^p \cup \mathcal{G}^c \cup \mathcal{G}_m^c, \tag{11}$$

where $\mathcal{E}_s = \mathcal{E}^p \cup \mathcal{E}_m^p \cup \mathcal{E}^c \cup \mathcal{E}_m^c$. Then, we have

$$QA^T + AQ + R^T B^T + BR \in \mathbb{S}_n^N(\mathcal{E}_s, 0).$$

Next, we build a chordal graph $\mathcal{G}_{ex} = (\mathcal{V}, \mathcal{E}_{ex})$ by making a chordal extension to graph $\mathcal{G}_s$. Define a graph $\mathcal{G}_0 = (\mathcal{V}, \mathcal{E}_0)$ which only contains nodes, but no edges. Then, (8) can be equivalently rewritten into (12),

$$\begin{cases} -(QA^T + AQ + R^T B^T + BR + \varepsilon I) \in \mathbb{S}_{n,+}^N(\mathcal{E}_{ex}, 0) \\ Q - \varepsilon I \in \mathbb{S}_{n,+}^N(\mathcal{E}_0, 0) \\ R \in \mathbb{R}_{m,n}^N(\mathcal{E}^c, 0) \end{cases} \tag{12}$$

where I is the identity matrix of appropriate dimension and $\varepsilon > 0$. As an example, Fig. 4 (a) presents a chordal graph $\mathcal{G}_{ex}$ for the hierarchical system shown in Fig. 3.

B. Decomposition of the Positive Semidefinite Constraints

After establishing the chordal characterization, we are now ready to apply Theorem 1 to decompose the positive semidefinite constraints in (12).

Let $\Gamma = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$ be the set of maximal cliques in graph $\mathcal{G}_{ex}$, and $\mathcal{T} = (\Gamma, \Xi)$ with $\Xi \subseteq \Gamma \times \Gamma$ be a clique tree.

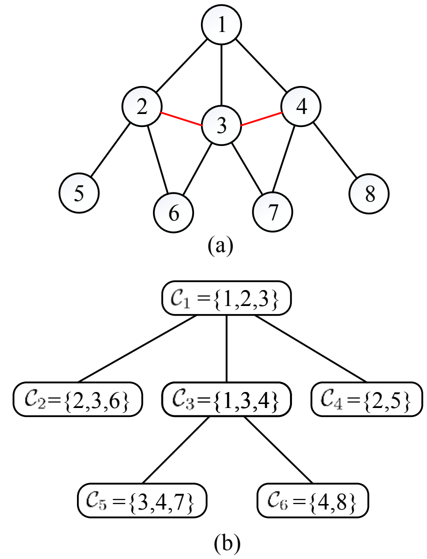

Fig. 4: Chordal extension and clique tree for Fig. 3: (a) chordal graph $\mathcal{G}_{ex}$, where two undirected edges (red ones) are added; (b) a clique tree.

The corresponding minimal set of overlapping elements is denoted by Λ. In (12), for notational simplicity, define

$$J_{Q,R} = -(QA^T + AQ + R^T B^T + BR + \varepsilon I).$$

According to Theorem 1, (12) is equivalent to

$$\begin{cases} \sum_{k=1}^{p} J_k = J_{Q,R}, \\ J_k \in \mathbb{S}_{n,+}^{\mathcal{C}_k}, k = 1, \ldots, p\ . \\ Q - \varepsilon I \in \mathbb{S}_{n,+}^N(\mathcal{E}_0, 0) \\ R \in \mathbb{R}_{m,n}^N(\mathcal{E}^c, 0) \end{cases} \tag{13}$$

The key feature in (13) is that it only involves a set of positive semidefinite constraints of small size (corresponding to the size of maximal cliques in $\mathcal{G}_{ex}$) instead of one large positive semidefinite constraint in (12). The price is that a large number of extra equality constraints are added in (13). In the next section, we further relax these constraints, resulting in a sequential design method.

C. Sequential Design Method over a Clique Tree

In this paper, the sequential design involves solving the feedback gains that only correspond to one maximal clique in $\mathcal{G}_{ex}$ each time. Both the order of this design sequence and information passing route depend on a clique tree that satisfies the running intersection property.

1) Basic ideas of the sequential design: The additional equality constraints in (13) only have impacts on the set of overlapping elements Λ in graph $\mathcal{G}_{ex}$. If Λ is empty, indicating the maximal cliques are disjoined, then the design of structured feedback gains for a large-scale system can be naturally decomposed into multiple sub-problems of small size. For the case in which there exist elements in Λ, our idea to decompose (13) is that we partition the coupling dynamic effect equally into several parts according to the maximal cliques which contain those overlapping elements.

2) Sequential design over a clique tree: Here, we introduce a formal description of the aforementioned strategy for decentralized systems over directed graphs.

Step 1: Get an averaging factor for overlapping elements

Given $\Gamma = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$ as the set of maximal cliques in graph $\mathcal{G}_{ex}$, we define $\gamma \in \mathbb{S}^N$ to characterize the number of repetitions of nodes and edges in Γ, *i.e.*,

$$\begin{cases} \gamma_{ii} = \text{the number of repetitions of node } i \text{ in } \Gamma \\ \gamma_{ij} = \text{the number of repetitions of edge } (i,j) \text{ in } \Gamma \end{cases}.$$

Note that $\gamma \in \mathbb{S}_1^N(\mathcal{E}_{ex}, 0)$. Correspondingly, we define an averaging factor $\gamma' \in \mathbb{S}_1^N(\mathcal{E}_{ex}, 0)$ for graph $\mathcal{G}_{ex}$ as follows:

$$\begin{cases} \gamma'_{ij} = \dfrac{1}{\gamma_{ij}} & \text{if } \gamma_{ij} \neq 0 \\ \gamma'_{ij} = 0 & \text{otherwise} \end{cases}.$$

Then, the averaging factor for the overlapping elements is defined as

$$\beta = \gamma' \otimes 1_{n\times n}, \tag{14}$$

where $1_{n\times n}$ is a matrix of dimension $n \times n$ with all entries as 1, and $\otimes$ denotes the Kronecker product.

Step 2: Derive a set of LMIs over maximal cliques

In this step, we *a priori* choose J_k in (13) as

$$J_k = J_{Q,R}(\mathcal{C}_k, \mathcal{C}_k) \circ \beta(\mathcal{C}_k, \mathcal{C}_k), k = 1, \ldots, p, \tag{15}$$

where $\circ$ denotes the Hadamard product.

Based on this construction, we have $\sum_{k=1}^{p} J_k = J_{Q,R}$. Thus, (13) is reduced into a set of small-size LMIs $\mathcal{L}_k, k = 1, \ldots, p$ over maximal cliques, in which each $\mathcal{L}_k$ is defined as

$$\mathcal{L}_k : \begin{cases} J_{Q,R}(\mathcal{C}_k, \mathcal{C}_k) \circ \beta(\mathcal{C}_k, \mathcal{C}_k) \succeq 0, \\ Q_j - \varepsilon I \succeq 0, j \in \mathcal{C}_k, \\ R(\mathcal{C}_k, \mathcal{C}_k) \in \mathbb{R}^N_{m,n}(\mathcal{E}^c, 0), \end{cases}. \tag{16}$$

Step 3: Sequential solution over a clique tree

The dimension of each LMI $\mathcal{L}_k$ corresponds to the size of maximal clique $\mathcal{C}_k$. There may exist some common design parameters among different $\mathcal{L}_k$. Due to the running intersection property of chordal graphs, we can sequentially solve them clique-by-clique over a clique tree $\mathcal{T}$.

Specifically, starting from the root clique in $\mathcal{T}$, we perform a tree traversal by embedding the overlapping parameters from cliques on the layer above. Breadth-first strategy is used in our simulation, which starts at the root, and explores the neighbour nodes first before moving to the next level. Take Fig. 4 as an example to demonstrate this idea. We first solve the LMI $\mathcal{L}_1$ corresponding to root clique $\mathcal{C}_1 = \{1, 2, 3\}$ to obtain the feedback gains in nodes $1, 2, 3$. Embedding these gains to the cliques in the second layer of the clique tree, *i.e.*, $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$, we can obtain the feedback gains corresponding to nodes $6, 4$ and 5, respectively.

V. Illustrative Examples

In this section, two illustrative examples are used to demonstrate the efficiency of the proposed sequential design method. All simulations were run on a computer with an Intel(R) Core(TM) i7 CPU, 2.8 GHz processor and 8GB of RAM. SeDuMi [26] was used.

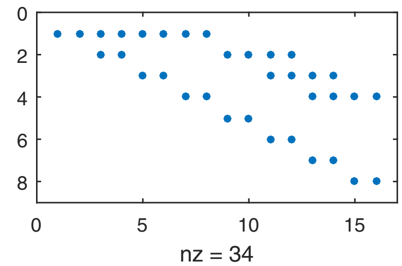

Fig. 5: Sparsity pattern of the structured feedback gains for the hierarchical system shown in Fig. 3.

TABLE I: Sequential solutions for the hierarchical system in Fig. 3

Seq.	Cliques	Computed gains	Time (s)
1	$\mathcal{C}_1$	$k_{11} = -\begin{bmatrix}30.83 & 7.26\end{bmatrix}, k_{12} = -\begin{bmatrix}1.64 & 1.30\end{bmatrix}$; $k_{13} = -\begin{bmatrix}1.19 & 0.98\end{bmatrix}, k_{22} = -\begin{bmatrix}9.05 & 5.88\end{bmatrix}$; $k_{33} = -\begin{bmatrix}9.99 & 6.28\end{bmatrix}$	0.0339
2	$\mathcal{C}_2$	$k_{66} = -\begin{bmatrix}6.75 & 4.51\end{bmatrix}, k_{26} = -\begin{bmatrix}0.08 & 0.13\end{bmatrix}$; $k_{36} = -\begin{bmatrix}0.06 & 0.25\end{bmatrix}$	0.0307
3	$\mathcal{C}_3$	$k_{44} = -\begin{bmatrix}9.15 & 5.77\end{bmatrix}$	0.0313
4	$\mathcal{C}_4$	$k_{55} = -\begin{bmatrix}6.64 & 4.41\end{bmatrix}, k_{25} = -\begin{bmatrix}0.12 & 0.23\end{bmatrix}$	0.0310
5	$\mathcal{C}_5$	$k_{77} = -\begin{bmatrix}6.74 & 4.50\end{bmatrix}, k_{37} = -\begin{bmatrix}0.03 & 0.13\end{bmatrix}$; $k_{47} = -\begin{bmatrix}0.04 & 0.29\end{bmatrix}$	0.0316
6	$\mathcal{C}_6$	$k_{88} = -\begin{bmatrix}6.57 & 4.32\end{bmatrix}, k_{48} = -\begin{bmatrix}0.01 & 0.15\end{bmatrix}$	0.0302

A. Hierarchical Systems

We first consider the hierarchical system in Fig. 3. Motivated by [12], we assume each node is an unstable second-order system coupled with its neighbouring nodes as follows:

$$\dot{x}_i = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x_i + \sum_{j \in \mathbb{N}_i^p} e^{-\alpha(i,j)} x_j + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \tag{17}$$

where, $\alpha(i,j)$ is chosen as $\frac{1}{10}(i-j)^2$ in the simulations. The feedback gains are in the form of (2). Fig. 5 shows the sparsity pattern of this controller.

We first solve this problem in a centralized way (*i.e.*, solving (8) directly) obtaining the following stabilizing controller:

- node 1: $k_{11} = -[22.3, 6.07], k_{12} = -[1.05, 1.09], k_{13} = -[0.77, 0.81], k_{14} = -[0.46, 0.50]$,
- node 2: $k_{22} = -[9.77, 4.88], k_{25} = -[0.25, 0.48], k_{26} = -[0.09, 0.24]$,
- node 3: $k_{33} = -[9.75, 4.84], k_{36} = -[0.23, 0.47], k_{37} = -[0.10, 0.23]$,
- node 4: $k_{44} = -[9.64, 4.79], k_{47} = -[0.23, 0.46], k_{48} = -[0.11, 0.23]$,
- node 5: $k_{55} = -[6.57, 4.32]$,
- node 6: $k_{66} = -[6.58, 4.34]$,
- node 7: $k_{77} = -[6.58, 4.34]$,
- node 8: $k_{88} = -[6.55, 4.29]$.

Then, we used the proposed sequential design approach to solve this problem (*i.e.*, solving (16) sequentially). The corresponding chordal extension and clique tree are shown in Fig. 4. TABLE I lists the solving sequences, computed gains and time consumed for each clique. We notice that for this special small-size problem, computing the gains in a centralized way was faster than that using the sequential method. However, it took less time for solving each maximal

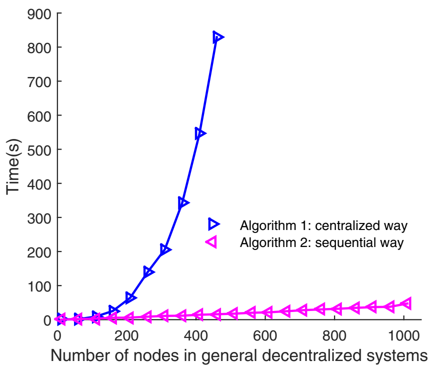

Fig. 6: Time in seconds of Centralized way (*i.e.*, solving (8)) versus Sequential way (*i.e.*, solving (16)) for solving a general decentralized control design problem with fixed size of largest maximal clique in $\mathcal{G}_{ex}$.

clique, as listed in TABLE I. The sequential design method would be beneficial for large-scale systems.

B. General Decentralized Systems

Here, we present simulation results for decentralized systems with general directed graphs. It is assumed that each node has the dynamics shown in (17). In the simulations, we first generate a random chordal graph $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ with a bound on the size of its largest maximal clique, and then randomly remove some edges of $\mathcal{G}_1$ to form the plant graph $\mathcal{G}^p$. To improve the feasibility, we ensure that the communication graph satisfies $\mathcal{E}^p \subseteq \mathcal{E}^c \subseteq \mathcal{E}_1$. This way, $\mathcal{G}^p, \mathcal{G}^c$ are general directed graphs such that the largest maximal clique of their chordal extension $\mathcal{G}_{ex}$ has limited size. When this is set to five, Fig. 6 shows a comparison between the performance of the centralized approach and sequential approach for different graph sizes. Using our sequential design approach, we could obtain stable structured feedback gains for a network of 1000 nodes within 50 s. However, using a centralized approach, we could not get a solution within 1000 s if the network size is over 500.

VI. CONCLUSION

This paper proposed a sequential design approach for the synthesis of static structured feedback gains by exploiting the chordal decomposition of block structured semidefinite matrices. We first presented a result on chordal decomposition that extended a result by Agler *et al.* [16] to the case of block matrices. Then, a simple relaxation technique was used, leading to a convex formulation that preserves the sparsity pattern of the original problem. Combining these two results, and using the running intersection property of chordal graphs, we further proposed a sequential design approach to solve structured feedback gains clique-by-clique over a clique tree. This method greatly improves the computational efficiency, demonstrated by two illustrative examples.

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