Convex Parameterization of Stabilizing Controllers and its LMI-based Computation via Filtering

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Outline

Introduction: stability and non-convexity

Closed-loop convexity and parameterizations

A filtering perspective and LMI formulation

Conclusions and future work

Stability and Stabilization

An autonomous system $x_{t+1} = Ax_t$ is *asymptotically stable* if and only if A is Schur stable, i.e.,

$$|\lambda_i(A)| < 1, \qquad i = 1, \dots n.$$

Stabilization

$$x_{t+1} = Ax_t + Bu_t$$
$$y_t = Cx_t$$

- Static state feedback $u_t = Kx_t$ stabilizes the system if and only if

$$|\lambda_i(A + BK)| < 1, \qquad i = 1, \dots n.$$

- Dynamical output feedback $\mathbf{u} = \mathbf{K}\mathbf{y}$ with

$$\begin{aligned} \xi_{t+1} &= A_{\mathsf{K}}\xi_t + B_{\mathsf{K}}y_t \\ u_t &= C_{\mathsf{K}}\xi_t + D_{\mathsf{K}}y_t \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x_{t+1} \\ \xi_{t+1} \end{bmatrix} = \begin{bmatrix} A + BD_{\mathsf{K}}C & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \begin{bmatrix} x_t \\ \xi_t \end{bmatrix},$$

  $\mathbf{K}$ stabilizes the system if and only if

$$\left| \lambda_i \left(\begin{bmatrix} A + BD_{\mathsf{K}}C & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \right) \right| < 1$$

Non-convexity

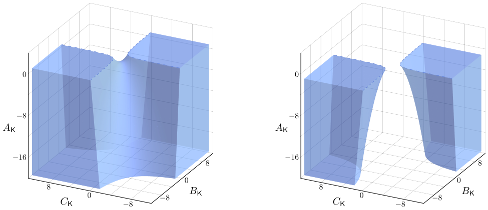
The set of stabilizing (static or dynamical) controllers is **non-convex**.

- **Static state feedback:** $u_t = Kx_t$

$$\mathcal{C}_1 = \{K \in \mathbb{R}^{m\times n} \mid A + BK \text{ is Schur stable}\}$$

- **Dynamical output feedback** $\mathbf{u} = \mathbf{K}\mathbf{y}$ with $\mathbf{K} = C_{\mathsf{K}}(zI - A_{\mathsf{K}})^{-1}B_{\mathsf{K}} + D_{\mathsf{K}}$

$$\mathcal{C}_2 = \{(A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}, D_{\mathsf{K}}) \mid \begin{bmatrix} A + BD_{\mathsf{K}}C & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \text{ is Schur stable}\}$$



(a) $\mathcal{C}_2$: Connected (b) $\mathcal{C}_2$: Disconnected [1]

[1] Figure from Tang, Y., Zheng, Y. & Li, N. (2021). Analysis of the optimization landscape of Linear Quadratic Gaussian (LQG) control. preprint arXiv:2102.04393.

Input-output responses

- Dynamics and controller $\mathbf{K} = C_{\mathsf{K}}(zI - A_{\mathsf{K}})^{-1}B_{\mathsf{K}} + D_{\mathsf{K}}$ with noises

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + \delta_{x_t}, \\ y_t &= Cx_t + \delta_{y_t}, \end{aligned} \quad + \quad \begin{aligned} \xi_{t+1} &= A_k\xi_t + B_k y_t, \\ u_t &= C_k\xi_t + D_k y_t + \delta_{u_t}, \end{aligned}$$

- Closed-loop responses from $(\boldsymbol{\delta}_x, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix},$$

where $\mathbf{\Phi}_{xx} = (zI - A - B\mathbf{K}C)^{-1}$ and

$$\begin{aligned} \mathbf{\Phi}_{xy} &= \mathbf{\Phi}_{xx}B\mathbf{K}, & \mathbf{\Phi}_{xu} &= \mathbf{\Phi}_{xx}B, \\ \mathbf{\Phi}_{yx} &= C\mathbf{\Phi}_{xx}, & \mathbf{\Phi}_{yy} &= C\mathbf{\Phi}_{xx}B\mathbf{K} + I, \\ \mathbf{\Phi}_{yu} &= C\mathbf{\Phi}_{xx}B, & \mathbf{\Phi}_{ux} &= \mathbf{K}C\mathbf{\Phi}_{xx}, \\ \mathbf{\Phi}_{uy} &= \mathbf{K}(C\mathbf{\Phi}_{xx}B\mathbf{K} + I), & \mathbf{\Phi}_{uu} &= \mathbf{K}C\mathbf{\Phi}_{xx}B + I. \end{aligned}$$

Closed-loop convexity: Enforcing stability becomes a "convex" constraint in certain closed-loop responses (Boyd & Barratt, 1991).

Closed-loop convexity and Today's Talk

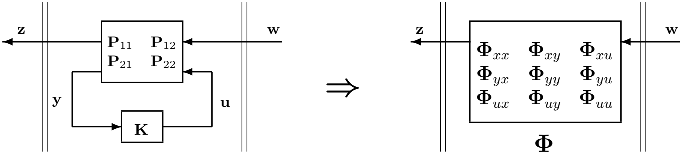


Figure: **Closed-loop optimization**: instead of optimizing control policies $\mathbf{K}$ (left), we directly optimize the closed-loop behavior $\mathbf{\Phi}$ (right).

Classical and recent approaches

- Youla parameterization (Youla, Jabr, and Bongiorno, 1976)
- System-level parameterization (Wang, Matni, and Doyle, 2019)
- Input-output parameterization (Furieri, Zheng, Papachristodoulou, and Kamgarpour, 2019)

Challenge: Despite being convex, closed-loop responses are infinitely dimensional.

- Finite-dimensional approximations (such as FIR truncation) are inefficient and impractical.
- No efficient numerical methods for computation!

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Closed-loop convexity

Closed-loop responses $$\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu}\\ \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix}\begin{bmatrix}\boldsymbol{\delta}_x\\ \boldsymbol{\delta}_y\\ \boldsymbol{\delta}_u\end{bmatrix},$$

Theorem (System-level parameterization (Wang, Matni, and Doyle, 2019))

Output-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ *stabilizes the system if and only if*

$$\begin{bmatrix}\mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy}\\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy}\end{bmatrix} \in \mathcal{RH}_\infty \quad \text{and} \quad \begin{bmatrix}zI - A & -B\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy}\\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy}\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix},$$

$$\begin{bmatrix}\mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy}\\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy}\end{bmatrix}\begin{bmatrix}zI - A\\ -C\end{bmatrix} = \begin{bmatrix}I\\ 0\end{bmatrix},$$

Theorem (Input-output parameterization (Furieri, et al., 2019))

Output-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ *stabilizes the system if and only if*

$$\begin{bmatrix}\mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix} \in \mathcal{RH}_\infty \quad \text{and} \quad \begin{bmatrix}I & -\mathbf{G}\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix},$$

$$\begin{bmatrix}\mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix}\begin{bmatrix}-\mathbf{G}\\ I\end{bmatrix} = \begin{bmatrix}0\\ I\end{bmatrix}.$$

Youla parameterization

A collection of stable transfer functions, $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r$ is called a doubly co-prime factorization of $\mathbf{G}$ if

$$\mathbf{G} = \mathbf{N}_r \mathbf{M}_r^{-1} = \mathbf{M}_l^{-1} \mathbf{N}_l$$

and

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = I.$$

Theorem (Youla parameterization (Youla, et al., 1976))

Output-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ *stabilizes the system if and only if*

$$\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} \qquad \textit{and} \qquad \mathbf{Q} \in \mathcal{RH}_\infty$$

A simple observation

- Define $\mathbf{X} = \mathbf{U}_r - \mathbf{N}_r \mathbf{Q}$, and $\mathbf{Y} = \mathbf{V}_r - \mathbf{M}_r \mathbf{Q}$. Then, we have

$$\mathbf{M}_l \mathbf{X} - \mathbf{N}_l \mathbf{Y} = I, \qquad \mathbf{X}, \mathbf{Y} \in \mathcal{RH}_\infty$$

A variant of Youla parameterization

Theorem

Output-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ *stabilizes the system if and only if*

$$\mathbf{K} = \mathbf{Y}\mathbf{X}^{-1} \qquad \text{and} \qquad \mathbf{M}_l\mathbf{X} - \mathbf{N}_l\mathbf{Y} = I, \qquad \mathbf{X}, \mathbf{Y} \in \mathcal{RH}_\infty$$

Two features:

- This parameterization only has one affine constraint.
- The equality does not need to hold exactly.

Theorem

Output-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ *stabilizes the system if and only if there exist* $\mathbf{X}$ *and* $\mathbf{Y}$ *in* $\mathcal{RH}_\infty$ *such that*

$$\|\mathbf{M}_l\mathbf{X} - \mathbf{N}_l\mathbf{Y} - I\|_\infty < 1. \qquad (1)$$

If (1) *holds, then* $\mathbf{K} = \mathbf{Y}\mathbf{X}^{-1}$ *internally stabilizes* $\mathbf{G}$.

Outline

A filtering perspective

- A *robust filtering* interpretation: find a stable filter $\begin{bmatrix}\mathbf{X} & \mathbf{Y}\end{bmatrix} \in \mathcal{RH}_\infty$ such that the residual $\mathbf{M}_l\mathbf{X} - \mathbf{N}_l\mathbf{Y} - I$ has $\mathcal{H}_\infty$ norm less than 1.

$$\|\mathbf{M}_l\mathbf{X} - \mathbf{N}_l\mathbf{Y} - I\|_\infty < 1$$

Right-filtering problem vs **Left-filtering problem**[2]

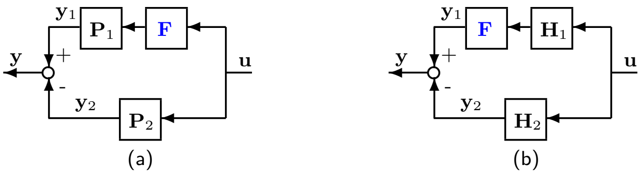

Figure: (a) Right-filtering problem (the filter $\mathbf{F}$ appears *before* the dynamical system $\mathbf{P}_1$). (b) Left-filtering problem (the filter $\mathbf{F}$ appears *after* the dynamical system $\mathbf{H}_1$)

- Classical literature on robust filtering focuses on the left-filtering problem.

[2] Geromel, Bernussou, Garcia, & de Oliveira (2000). $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Robust Filtering for Discrete-Time Linear Systems. SIAM Journal on Control and Optimization, 38(5), 1353-1368.

A filtering perspective

Right $\mathcal{H}_\infty$ filtering problem: given $\mu > 0$ and $\mathbf{P}_1(z), \mathbf{P}_2(z) \in \mathcal{RH}_\infty$ with a state-space realization

$$\begin{bmatrix} \mathbf{P}_1(z) & \mathbf{P}_2(z) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right],$$

find a stable filter $\mathbf{F}(z) \in \mathcal{RH}_\infty$ such that

$$\|\mathbf{P}_1(z)\mathbf{F}(z) - \mathbf{P}_2(z)\|_\infty < \mu. \tag{2}$$

Lemma (KYP lemma)

Let $\mathbf{T}(z) = C(zI - A)^{-1}B + D \in \mathcal{RH}_\infty$. $\|\mathbf{T}(z)\|_\infty^2 < \mu$ *if and only if*

$$\begin{bmatrix} P & AP & B & 0 \\ PA^\mathsf{T} & P & 0 & PC^\mathsf{T} \\ B^\mathsf{T} & 0 & I & D^\mathsf{T} \\ 0 & CP & D & \mu I \end{bmatrix} \succ 0, \qquad P \succ 0.$$

Decentralized filter/control

Theorem

There exists $\mathbf{F}(z) \in \mathcal{RH}_\infty$ *such that* (2) *holds if and only if*

$$\begin{bmatrix} X & Z & AX+B_1L & AZ+B_1L & B_1R-B_2 & 0 \\ \star & Z & Q & Q & F & 0 \\ \star & \star & X & Z & 0 & XC^\mathsf{T}+L^\mathsf{T}D_1^\mathsf{T} \\ \star & \star & \star & Z & 0 & ZC^\mathsf{T}+L^\mathsf{T}D_1^\mathsf{T} \\ \star & \star & \star & \star & I & R^\mathsf{T}D_1^\mathsf{T}-D_2^\mathsf{T} \\ \star & \star & \star & \star & \star & \mu^2 I \end{bmatrix} \succ 0.$$

A state-space realization of the filter $\mathbf{F}(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D}$ is

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} U^\mathsf{T} Z^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q & F \\ L & R \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix}^\mathsf{T},$$

where U is any non-singular matrix (represents a similarity transformation).

- The order of the filter is the same as the system dynamics $\boldsymbol{P}_1$ (i.e., full-order filter).
- Imposing block-diagonal structures on Z, Q, F, L, R leads to a decentralized filter.

Decentralized filter/control

Theorem

Let $\mathbf{M}_l$ and $\mathbf{N}_l$ have the state-space realization

$$\begin{bmatrix}\mathbf{M}_l(z) & \mathbf{N}_l(z)\end{bmatrix} = \left[\begin{array}{c|cc} A & B_M & B_N \\ \hline C & D_M & D_N \end{array}\right].$$

There exist $\mathbf{X}(z)$ and $\mathbf{Y}(z)$ in $\mathcal{RH}_\infty$ such that

$$\|\mathbf{M}_l(z)\mathbf{X}(z) - \mathbf{N}_l(z)\mathbf{Y}(z) - I\|_\infty < \epsilon$$

if and only if the following LMI is feasible

$$\begin{bmatrix} X & Z & f_1(X, L_X, L_Y) & f_2(Z, L_X, L_Y) & f_3(R_X, R_Y) & 0 \\ \star & Z & Q & Q & F & 0 \\ \star & \star & X & Z & 0 & f_4(X, L_X, L_Y) \\ \star & \star & \star & Z & 0 & f_5(Z, L_X, L_Y) \\ \star & \star & \star & \star & I & f_6(R_X, R_Y) \\ \star & \star & \star & \star & \star & \epsilon^2 I \end{bmatrix} \succ 0. \quad (3)$$

Furthermore, the state-space realizations for $\mathbf{X}(z)$ and $\mathbf{Y}(z)$, as well as $\mathbf{K} = \mathbf{Y}\mathbf{X}^{-1}$, have the same order as the plant.

Numerical experiments

Decentralized stabilization

$$\begin{aligned} \min \quad & h(Q, F, L_X, L_Y, R_X, R_Y) \\ \text{subject to} \quad & (3), \\ & X \succ 0, \quad Z, Q, F \text{ block diagonal}, \\ & L_X, L_Y, R_X, R_Y \text{ block diagonal}, \end{aligned} \tag{4}$$

where $h(R, F, L_X, L_Y, R_X, R_Y) :=$
$\|Q\|_\infty + \|F\|_\infty + \|L_X\|_\infty + \|L_Y\|_\infty + \|R_X\|_\infty + \|R_Y\|_\infty$

Example

Consider a chain of second-order subsystems with dynamics

$$x_i[t+1] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} x_i[t] + \sum_{j \in \mathcal{N}_i} \alpha(i,j) x_j[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i[t],$$

$$y_i[t] = \begin{bmatrix} 0 & 1 \end{bmatrix} x_i[t],$$

where $\alpha(i,j) = \frac{1}{5} e^{-(i-j)^2}$, $\mathcal{N}_i = \{i-1, i+1\} \cap \{1, \dots, n\}$ and $i = 1, \dots, n$.

Numerical experiments

Case 1: three subsystems $n = 3$

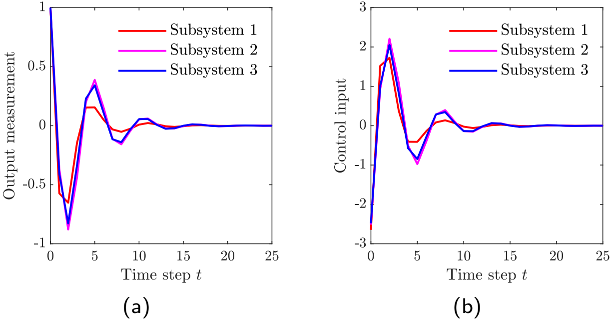

Figure: Responses with three subsystems $n = 3$: (a) Output measurement $y_i[t]$; (b) Input $u_i[t]$.

Numerical experiments

Case 2: varying the number of subsystems, and comparison with SLP+FIR

- **Efficient computation**: Order of magnitudes faster in time consumption

Table: Time (in seconds) for (4) and SLP + FIR (length 20). Includes YALMIP time and MOSEK time.

# of nodes n	6	8	10	12	14
LMI (4)	0.49	0.60	0.75	0.99	1.28
SLP + FIR	3.22	8.60	22.68	53.19	132.87

- **Efficient implementation:** Controller order does not increase with the length of FIR approximation.

Table: Controller order for (4) and SLP + FIR (length 20).

# of nodes n	6	8	10	12	14
LMI (4)	2	2	2	2	2
SLP + FIR	468	624	780	936	1092

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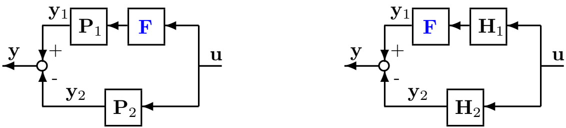
Summary

Stability is a non-convex constraint.

- Closed-loop parameterization (Youla/SLP/IOP etc) is convex but infinite dimensional.
- A variant of Youla parameterization has a single affine constraint

$$\|\mathbf{M}_l \mathbf{X} - \mathbf{N}_l \mathbf{Y} - I\|_\infty < 1$$

which has an interesting robust filtering interpretation



(a) (b)

- This leads to an efficient LMI to parameterize all full-order stabilizing controllers (*efficient computation and implementation*!).

Future work:

- Simultaneous filtering and Optimal control

Thank you for your attention!

Q & A

de Oliveira, Mauricio C., and Yang Zheng. "Convex Parameterization of Stabilizing Controllers and its LMI-based Computation via Filtering." arXiv preprint arXiv:2203.17145 (2022).