Sample Complexity of Model-based Linear Quadratic Control: LQR and LQG

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Main references

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- Zheng, Y., Furieri, L., Kamgarpour, M., & Li, N. (2021, May). Sample complexity of linear quadratic gaussian (LQG) control for output feedback systems. In Learning for Dynamics and Control (pp. 559-570). PMLR.
- Zheng, Y., Furieri, L., Kamgarpour, M., & Li, N. (2022). System-level, input–output and new parameterizations of stabilizing controllers, and their numerical computation. Automatica, 140, 110211.

Outline

Motivation and problem formulation

Stability and convexity in frequency-domain

Robust formulation of LQR and its suboptimality

Robust formulation of LQG and its suboptimality

Summary

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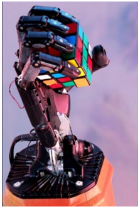
Motivation

Data-driven control (model free vs model-based)

- Become very popular in both academia and practice,
- Impressive empirical results in many applications: from game playing, robotics, and drones, etc.

(a) DeepMind


(b) Open-AI


(c) Applications

Challenges: Lack of non-asymptotic performance guarantees

- Sample complexity
- Suboptimality
- Robustness, etc.

Today's lecture: Optimal control

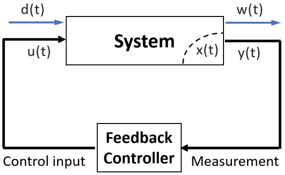


Figure: Feedback paradigm

Linear Quadratic Optimal control

$$\min_{u_1,u_2,\ldots,} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^{\mathsf{T}}Qx_t + u_t^{\mathsf{T}}Ru_t\right)\right]$$

subject to $\quad x_{t+1} = Ax_t + Bu_t + w_t$

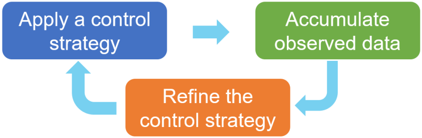
$\qquad y_t = Cx_t + v_t$

- Many practical applications
- **Linear Quadratic Regulator (LQR)** when the state x_t directly observable
- **Linear Quadratic Gaussian (LQG)** control when only y_t is observed
- Extensive classical results (Dynamic programming, Separation principle, Riccati equations, etc.)

Major challenge: how to perform optimal control when the system is unknown?

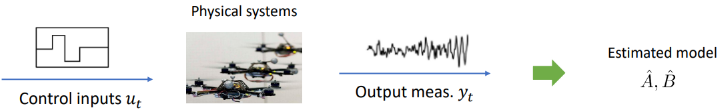
Model free vs Model based approaches

- **Model-free policy optimization**



  – Analysis is mainly based on geometrical landscape, stationary points, saddle points, smoothness constants, convergence etc.

- **Model-based certainty-equivalence or robust control**



  – Certainty-equivalence control treats the estimated model as the truth
  – **Robust control** explicitly takes into account the estimation error
  – **Perturbation analysis**

Model-based LQR

- **Standard Linear Quadratic Regulator**

$$J^* := \min_{u_1,u_2,\ldots,} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = Ax_t + Bu_t + w_t$$

  - Optimal policy is static $u_t = Kx_t$, e.g., from the Ricatti equation

- **Robust LQR**

$$J^*(\epsilon_A,\epsilon_B) := \min_{u_1,u_2,\ldots,} \sup_{\|\Delta_A\|\leq\epsilon_A, \|\Delta_B\|\leq\epsilon_B} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = (A+\Delta_A)x_t + (B+\Delta_B)u_t + w_t$$

- These two problems become the same when $\max\{\epsilon_A,\epsilon_B\} \to 0$.
- It is unclear
  - how $J^*(\epsilon_A,\epsilon_B) - J^*$ changes as $\max\{\epsilon_A,\epsilon_B\} \to 0$.
  - how to achieve $J^*(\epsilon_A,\epsilon_B)$? The optimal policy form: static, dynamic, nonlinear, existence etc?

Model-based LQG

- **Standard LQG**

$$J^* := \min_{u_1, u_2, \ldots,} \quad \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^{\mathsf{T}} Q x_t + u_t^{\mathsf{T}} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = Ax_t + Bu_t + w_t$$

$$y_t = Cx_t + v_t$$

  - Optimal policy is a dynamical controller $\mathbf{u} = \mathbf{K}\mathbf{y}$, e.g., from two Ricatti equations

$$\hat{x}_{t+1} = (A - BK)\hat{x}_t + L(y_t - C\hat{x}_t),$$

$$u_t = -K\hat{x}_t.$$

- **Robust LQG** - different from LQR
  - Estimation of $\hat{A}, \hat{B}, \hat{C}$ in state-space is not unique
  - The representation $\mathbf{G} = C(zI - A)^{-1}B$ in frequency domain is unique.
  - We estimate a transfer function $\hat{\mathbf{G}}$ as well as its uncertainty (open-loop stable systems)

$$\|\Delta\|_\infty := \|\mathbf{G} - \hat{\mathbf{G}}\|_\infty < \epsilon$$

Model-based LQG

- **Modified LQG**

$$
\begin{aligned}
J^* := \min_{u_1, u_2, \dots,} \quad & \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t\right)\right] \\
\text{subject to} \quad & x_{t+1} = Ax_t + Bu_t + Bw_t \\
& y_t = Cx_t + v_t
\end{aligned}
$$

- **Robust LQG**

$$
\begin{aligned}
J^*(\epsilon) := \min_{\mathbf{K}} \sup_{\|\mathbf{\Delta}\|_\infty < \epsilon} \quad & \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t\right)\right], \\
\text{subject to} \quad & \mathbf{y} = (\hat{\mathbf{G}} + \mathbf{\Delta})\mathbf{u} + \mathbf{v} \\
& \mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{w},
\end{aligned}
$$

- These two problems become the same when $\epsilon \to 0$.
- It is unclear
  - how $J^*(\epsilon) - J^*$ changes as $\epsilon \to 0$.
  - how to achieve $J^*(\epsilon)$?

Outline

Motivation and problem formulation

Stability and convexity in frequency-domain

Robust formulation of LQR and its suboptimality

Robust formulation of LQG and its suboptimality

Summary

Stability and Stabilization

An autonomous system $x_{t+1} = Ax_t$ is *asymptotically stable* if and only if A is Schur stable, i.e.,

$$|\lambda_i(A)| < 1, i = 1, \dots n.$$

Stabilization

$$x_{t+1} = Ax_t + Bu_t$$
$$y_t = Cx_t$$

- Static state feedback $u_t = Kx_t$ stabilizes the system if and only if

$$|\lambda_i(A + BK)| < 1, i = 1, \dots n.$$

- Dynamical output feedback $\mathbf{u} = \mathbf{K}\mathbf{y}$ with

$$\begin{aligned} \xi_{t+1} &= A_{\mathsf{K}}\xi_t + B_{\mathsf{K}}y_t \\ y_t &= C_{\mathsf{K}}\xi_t \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x_{t+1} \\ \xi_{t+1} \end{bmatrix} = \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \begin{bmatrix} x_t \\ \xi_t \end{bmatrix},$$

  stabilizes the system if and only if

$$\left|\lambda_i\left(\begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix}\right)\right| < 1$$

Frequency-domain characterization

- Dynamics and controller with noises

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + \delta_{x_t}, \\ y_t &= Cx_t + \delta_{y_t}, \end{aligned} \quad + \quad \begin{aligned} \xi_{t+1} &= A_k\xi_t + B_k y_t, \\ u_t &= C_k\xi_t + D_k y_t + \delta_{u_t}, \end{aligned}$$

- Closed-loop responses from $(\boldsymbol{\delta}_x, \boldsymbol{\delta}_y, \boldsymbol{\delta}_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{xx} & \boldsymbol{\Phi}_{xy} & \boldsymbol{\Phi}_{xu} \\ \boldsymbol{\Phi}_{yx} & \boldsymbol{\Phi}_{yy} & \boldsymbol{\Phi}_{yu} \\ \boldsymbol{\Phi}_{ux} & \boldsymbol{\Phi}_{uy} & \boldsymbol{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_x \\ \boldsymbol{\delta}_y \\ \boldsymbol{\delta}_u \end{bmatrix},$$

where $\boldsymbol{\Phi}_{xx} = (zI - A - B\mathbf{K}C)^{-1}$ and

$$\begin{aligned} \boldsymbol{\Phi}_{xy} &= \boldsymbol{\Phi}_{xx}B\mathbf{K}, & \boldsymbol{\Phi}_{xu} &= \boldsymbol{\Phi}_{xx}B, \\ \boldsymbol{\Phi}_{yx} &= C\boldsymbol{\Phi}_{xx}, & \boldsymbol{\Phi}_{yy} &= C\boldsymbol{\Phi}_{xx}B\mathbf{K} + I, \\ \boldsymbol{\Phi}_{yu} &= C\boldsymbol{\Phi}_{xx}B, & \boldsymbol{\Phi}_{ux} &= \mathbf{K}C\boldsymbol{\Phi}_{xx}, \\ \boldsymbol{\Phi}_{uy} &= \mathbf{K}(C\boldsymbol{\Phi}_{xx}B\mathbf{K} + I), & \boldsymbol{\Phi}_{uu} &= \mathbf{K}C\boldsymbol{\Phi}_{xx}B + I. \end{aligned}$$

Frequency-domain characterization

LQR Performance specification

- LQR cost: assuming $Q = I, R = I$ and the covariances of noises are I.

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t\right)\right] = \left\|\begin{bmatrix}\mathbf{\Phi}_{xx}\\ \mathbf{\Phi}_{ux}\end{bmatrix}\right\|_{\mathcal{H}_2}^2$$

- The standard LQR problem becomes

$$\begin{aligned}\min_{u_1,u_2,\ldots,} \quad & \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t\right)\right]\\ \text{subject to} \quad & x_{t+1} = Ax_t + Bu_t + w_t\end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned}\min_{\mathbf{K}} \quad & \left\|\begin{bmatrix}\mathbf{\Phi}_{xx}\\ \mathbf{\Phi}_{ux}\end{bmatrix}\right\|_{\mathcal{H}_2}^2\\ \text{subject to} \quad & \mathbf{u} = \mathbf{K}\mathbf{x}.\end{aligned} \quad \Leftrightarrow \quad \begin{aligned}\min_{\mathbf{K}} \quad & \left\|\begin{bmatrix}\mathbf{\Phi}_{xx}\\ \mathbf{\Phi}_{ux}\end{bmatrix}\right\|_{\mathcal{H}_2}^2\\ \text{subject to} \quad & \mathbf{K} \text{ stabilizes the system.}\end{aligned}$$

Frequency-domain characterization

LQG performance specification

- LQG cost: assuming $Q = I, R = I$ and the covariances of noises are I.

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t\right)\right] = \left\|\begin{bmatrix}\mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix}\right\|_{\mathcal{H}_2}^2$$

- The LQG problem becomes

$$\begin{aligned}\min_{u_1,u_2,\ldots,} \quad & \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t\right)\right]\\ \text{subject to} \quad & x_{t+1} = Ax_t + Bu_t + Bw_t\\ & y_t = Cx_t + v_t\end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned}\min_{\mathbf{K}} \quad & \left\|\begin{bmatrix}\mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix}\right\|_{\mathcal{H}_2}^2\\ \text{subject to} \quad & \mathbf{u} = \mathbf{K}\mathbf{y}.\end{aligned} \quad \Leftrightarrow \quad \begin{aligned}\min_{\mathbf{K}} \quad & \left\|\begin{bmatrix}\mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu}\\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu}\end{bmatrix}\right\|_{\mathcal{H}_2}^2\\ \text{subject to} \quad & \mathbf{K} \text{ stabilizes the system.}\end{aligned}$$

Convex reformulation

Closed-loop convexity: instead of optimizing over the controller $\mathbf{K}$, we directly optimize the closed-loop responses.

Theorem (System-level parameterization (SLP))

State-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{x}$ *stabilizes the system if and only if*

$$\begin{bmatrix} \mathbf{\Phi}_{xx} \\ \mathbf{\Phi}_{ux} \end{bmatrix} \in \mathcal{RH}_\infty \qquad \textit{and} \qquad \begin{bmatrix} zI - A & B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{xx} \\ \mathbf{\Phi}_{ux} \end{bmatrix} = I.$$

Theorem (Input-output parameterization (IOP))

Output-feedback controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ *stabilizes the system if and only if*

$$\begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \in \mathcal{RH}_\infty \qquad \textit{and} \qquad \begin{aligned} \begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} &= \begin{bmatrix} I & 0 \end{bmatrix}, \\ \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} &= \begin{bmatrix} 0 \\ I \end{bmatrix}. \end{aligned}$$

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Summary

Robust LQR

$$J^*(\epsilon_A, \epsilon_B) := \min_{u_1, u_2, \ldots,} \sup_{\|\Delta_A\| \le \epsilon_A, \|\Delta_B\| \le \epsilon_B} \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = (A + \Delta_A)x_t + (B + \Delta_B)u_t + w_t$$

- It is unclear how $J^*(\epsilon_A, \epsilon_B) - J^*$ changes as $\epsilon = \max\{\epsilon_A, \epsilon_B\} \to 0$.
- It is unclear how to achieve $J^*(\epsilon_A, \epsilon_B)$? The optimal policy form: static, dynamic, nonlinear, existence etc?

A sequence of inner approximations and upper bounds

- Design a dynamical controller $\mathbf{u} = \mathbf{K}\mathbf{x}$ via convex optimization such that

$$\frac{J(A, B, \mathbf{K}) - J^*}{J^*} \le \mathcal{O}(\epsilon), \qquad \text{when } \epsilon \text{ is small enough}$$

- Together with a standard OLS estimation of A, B, we derive an end-to-end sample complexity bound

$$\frac{J(A, B, \mathbf{K}) - J^*}{J^*} \le \mathcal{O}\left(C_{\mathrm{LQR}}\sqrt{\frac{(n+p)\log 1/\delta}{N}}\right)$$

provided that the number of samples N is large enough.

Robust LQR - upper bounds

- **Inner approximation 1**: dynamical controller

$$\mathbf{u} = \mathbf{K}\mathbf{x}$$

- **Inner approximation 2**: robust stabilization for all $A + \Delta_A, B + \Delta_B$ with $\|\Delta_A\| \leq \epsilon_A$ and $\|\Delta_B\| \leq \epsilon_B$

$$\|\mathbf{\Delta}\|_\infty := \left\| \begin{bmatrix} \Delta_A & \Delta_B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \right\|_\infty \leq \left\| \begin{bmatrix} \frac{\epsilon_A}{\sqrt{\alpha}} \mathbf{\Phi}_x \\ \frac{\epsilon_B}{\sqrt{1-\alpha}} \mathbf{\Phi}_u \end{bmatrix} \right\|_\infty < 1$$

where $\alpha \in (0, 1)$ is any fixed constant between 0 and 1.

- **Inner approximation 3**: upper bounds on the performance

$$\begin{aligned} J(A, B, \mathbf{K}) &= \left\| \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \left(I + \begin{bmatrix} \Delta_A & \Delta_B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \right)^{-1} \right\|_{\mathcal{H}_2} \\ &\leq \|(I + \mathbf{\Delta})^{-1}\|_\infty \left\| \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \right\|_{\mathcal{H}_2} \\ &\leq \frac{1}{1 - \|\mathbf{\Delta}\|_\infty} J(\hat{A}, \hat{B}, \mathbf{K}) \end{aligned}$$

Robust LQR - quasi convex optimization

- **Another upper bound** by combining approximation 1 & approximation 3

$$H_\alpha(\mathbf{\Phi}_x, \mathbf{\Phi}_u) := \left\| \begin{bmatrix} \frac{\epsilon_A}{\sqrt{\alpha}} \mathbf{\Phi}_x \\ \frac{\epsilon_B}{\sqrt{1-\alpha}} \mathbf{\Phi}_u \end{bmatrix} \right\|_\infty \quad \Rightarrow \quad J(A, B, \mathbf{K}) \leq \frac{\left\| \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \right\|_{\mathcal{H}_2}}{1 - H_\alpha(\mathbf{\Phi}_x, \mathbf{\Phi}_u)}$$

- Finally, we arrive at the following quasi-convex optimization formulation

$$\begin{aligned} \text{minimize}_{\gamma \in [0,1)} \frac{1}{1-\gamma} \min_{\mathbf{\Phi}_x, \mathbf{\Phi}_u} & \left\| \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \right\|_{\mathcal{H}_2} \\ \text{s.t.} & \begin{bmatrix} zI - \widehat{A} & -\widehat{B} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} = I, \quad \left\| \begin{bmatrix} \frac{\epsilon_A}{\sqrt{\alpha}} \mathbf{\Phi}_x \\ \frac{\epsilon_B}{\sqrt{1-\alpha}} \mathbf{\Phi}_u \end{bmatrix} \right\|_{\mathcal{H}_\infty} \leq \gamma \\ & \mathbf{\Phi}_x, \mathbf{\Phi}_u \in \frac{1}{z} \mathcal{RH}_\infty. \end{aligned} \tag{3.18}$$

Some remarks

- The controller is constructed as $\mathbf{K} = \mathbf{\Phi}_u \mathbf{\Phi}_x^{-1}$;
- The problem is still infinitely dimensional. An finite impulse response (FIR) approximation was used in the original paper.
- *This might be inefficient in both computation and implementation.*

Robust LQR - suboptimality

Theorem 4.1. *Let $J_\star$ denote the minimal LQR cost achievable by any controller for the dynamical system with transition matrices (A, B), and let $K_\star$ denote the optimal contoller. Let $(\widehat{A}, \widehat{B})$ be estimates of the transition matrices such that $\|\Delta_A\|_2 \le \epsilon_A$, $\|\Delta_B\|_2 \le \epsilon_B$. Then, if $\mathbf{K}$ is synthesized via* (3.18) *with $\alpha = 1/2$, the relative error in the LQR cost is*

$$\frac{J(A, B, \mathbf{K}) - J_\star}{J_\star} \le 5(\epsilon_A + \epsilon_B \|K_\star\|_2)\|\mathfrak{R}_{A+BK_\star}\|_{\mathcal{H}_\infty} , \tag{4.1}$$

as long as $(\epsilon_A + \epsilon_B \|K_\star\|_2)\|\mathfrak{R}_{A+BK_\star}\|_{\mathcal{H}_\infty} \le 1/5$.

Some remarks

- The proof is essentially based on some careful perturbation analysis (first-order Taylor expansion)
- Combining with standard OLS estimation of A, B, one can show (Corollary 4.3)

$$\frac{J(A, B, \mathbf{K}) - J^*}{J^*} \le \mathcal{O}\left(C_{\mathrm{LQR}} \sqrt{\frac{(n+p)\log 1/\delta}{N}}\right)$$

provided that the number of samples N is large enough.

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Summary

Robust LQG

$$J^*(\epsilon) := \min_{\mathbf{K}} \sup_{\|\mathbf{\Delta}\|_\infty < \epsilon} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t\right)\right],$$

$$\text{subject to} \quad \mathbf{y} = (\hat{\mathbf{G}} + \mathbf{\Delta})\mathbf{u} + \mathbf{v}$$

$$\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{w},$$

- It is unclear how $J^*(\epsilon) - J^*$ changes as $\epsilon \to 0$.
- It is unclear how to achieve $J^*(\epsilon)$?

One upper bound

- Design a dynamical controller $\mathbf{u} = \mathbf{K}\mathbf{y}$ via convex optimization such that

$$\frac{\hat{J} - J_\star}{J_\star} \leq \mathcal{O}(\epsilon), \qquad \text{when } \epsilon \text{ is small enough}$$

- Together with a standard OLS estimation of $\mathbf{G}$, we derive (open-loop stable systems)

$$\frac{\hat{J} - J_\star}{J_\star} \leq \mathcal{O}\left(\sqrt{\frac{T}{N}} + \rho(A_\star)^T\right),$$

with high probability provided the number of samples N is sufficiently large, where T is the length of FIR model estimation,

Robust LQG - equivalent formulation

Closed-loop convexity: Instead of designing $\mathbf{u} = \mathbf{K}\mathbf{y}$, we consider

$$\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}(I-\mathbf{G}_\star\mathbf{K})^{-1} & (I-\mathbf{G}_\star\mathbf{K})^{-1}\mathbf{G}_\star \\ \mathbf{K}(I-\mathbf{G}_\star\mathbf{K})^{-1} & (I-\mathbf{K}\mathbf{G}_\star)^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = \begin{bmatrix}\mathbf{Y} & \mathbf{W}\\ \mathbf{U} & \mathbf{Z}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix}$$

Theorem

The robust LQG problem is equivalent to

$$\min_{\hat{\mathbf{Y}},\hat{\mathbf{W}},\hat{\mathbf{U}},\hat{\mathbf{Z}}} \max_{\|\mathbf{\Delta}\|_\infty<\epsilon} \quad J(\mathbf{G}_\star,\mathbf{K}) = \left\| \begin{bmatrix}\hat{\mathbf{Y}}(I-\mathbf{\Delta}\hat{\mathbf{U}})^{-1} & \hat{\mathbf{Y}}(I-\mathbf{\Delta}\hat{\mathbf{U}})^{-1}(\hat{\mathbf{G}}+\mathbf{\Delta}) \\ \hat{\mathbf{U}}(I-\mathbf{\Delta}\hat{\mathbf{U}})^{-1} & (I-\hat{\mathbf{U}}\mathbf{\Delta})^{-1}\hat{\mathbf{Z}}\end{bmatrix}\right\|_{\mathcal{H}_2}$$

$$\text{subject to} \quad \begin{bmatrix}I & -\hat{\mathbf{G}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{Y}} & \hat{\mathbf{W}}\\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix},$$

$$\begin{bmatrix}\hat{\mathbf{Y}} & \hat{\mathbf{W}}\\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix}\begin{bmatrix}-\hat{\mathbf{G}}\\ I\end{bmatrix} = \begin{bmatrix}0\\ I\end{bmatrix},$$

$$\hat{\mathbf{Y}},\hat{\mathbf{W}},\hat{\mathbf{U}},\hat{\mathbf{Z}} \in \mathcal{RH}_\infty,\ \|\hat{\mathbf{U}}\|_\infty \le \frac{1}{\epsilon},$$

where the optimal robust controller is recovered as $\mathbf{K} = \hat{\mathbf{U}}\hat{\mathbf{Y}}^{-1}$.

Robust LQG - upper bound

$$J(\mathbf{G}_\star, \mathbf{K})^2 = \|\hat{\mathbf{Y}}(I - \boldsymbol{\Delta}\hat{\mathbf{U}})^{-1}\|_{\mathcal{H}_2}^2 + \|\hat{\mathbf{U}}(I - \boldsymbol{\Delta}\hat{\mathbf{U}})^{-1}\|_{\mathcal{H}_2}^2 + \|(I - \hat{\mathbf{U}}\boldsymbol{\Delta})^{-1}\hat{\mathbf{Z}}\|_{\mathcal{H}_2}^2 + \|\hat{\mathbf{Y}}(I - \boldsymbol{\Delta}\hat{\mathbf{U}})^{-1}(\hat{\mathbf{G}} + \boldsymbol{\Delta})\|_{\mathcal{H}_2}^2.$$

Proposition

If $\|\hat{\mathbf{U}}\|_\infty \leq \frac{1}{\epsilon}, \|\boldsymbol{\Delta}\|_\infty < \epsilon$ and $\hat{\mathbf{G}} \in \mathcal{RH}_\infty$, then, we have ($\hat{\mathbf{W}} = \hat{\mathbf{Y}}\hat{\mathbf{G}}$)

$$\|\hat{\mathbf{Y}}(I - \boldsymbol{\Delta}\hat{\mathbf{U}})^{-1}(\hat{\mathbf{G}} + \boldsymbol{\Delta})\|_{\mathcal{H}_2} \leq \frac{\|\hat{\mathbf{W}}\|_{\mathcal{H}_2} + \epsilon\|\hat{\mathbf{Y}}\|_{\mathcal{H}_2}(2 + \|\hat{\mathbf{U}}\|_\infty\|\hat{\mathbf{G}}\|_\infty)}{1 - \epsilon\|\hat{\mathbf{U}}\|_\infty}.$$

Theorem

If $\|\hat{\mathbf{U}}\|_\infty \leq \frac{1}{\epsilon}, \|\boldsymbol{\Delta}\|_\infty < \epsilon$ and $\hat{\mathbf{G}} \in \mathcal{RH}_\infty$, the robust LQG cost is upper bounded by

$$J(\mathbf{G}_\star, \mathbf{K}) \leq \frac{1}{1 - \epsilon\|\hat{\mathbf{U}}\|_\infty} \left\| \begin{bmatrix} \sqrt{1 + h(\epsilon, \|\hat{\mathbf{U}}\|_\infty)}\hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} \right\|_{\mathcal{H}_2}, \qquad (1)$$

where $\hat{\mathbf{Y}}, \hat{\mathbf{W}}, \hat{\mathbf{U}}, \hat{\mathbf{Z}}$ satisfy the IOP constraints, and the factor

$$h(\epsilon, \|\hat{\mathbf{U}}\|_\infty) = \mathcal{O}(\epsilon)$$

Robust LQG - quasi-convex optimization

Theorem

Given $\hat{\mathbf{G}} \in \mathcal{RH}_\infty$, a model estimation error ϵ, and any constant $\alpha > 0$, the robust LQG problem is upper bounded by the following problem

$$
\begin{aligned}
\min_{\gamma \in [0, 1/\epsilon)} \frac{1}{1 - \epsilon\gamma} \quad \min_{\hat{\mathbf{Y}}, \hat{\mathbf{W}}, \hat{\mathbf{U}}, \hat{\mathbf{Z}}} & \left\| \begin{bmatrix} \sqrt{1 + h(\epsilon, \alpha)}\hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} \right\|_{\mathcal{H}_2} \\
\textit{subject to} \quad & \begin{bmatrix} I & -\hat{\mathbf{G}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \\
& \begin{bmatrix} \hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}} \end{bmatrix} \begin{bmatrix} -\hat{\mathbf{G}} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\
& \hat{\mathbf{Y}}, \hat{\mathbf{W}}, \hat{\mathbf{Z}} \in \mathcal{RH}_\infty, \ \|\hat{\mathbf{U}}\|_\infty \leq \gamma, \|\hat{\mathbf{U}}\|_\infty \leq \alpha,
\end{aligned}
\tag{2}
$$

where $h(\epsilon, \alpha) = \epsilon \|\hat{\mathbf{G}}\|_\infty (2 + \alpha \|\hat{\mathbf{G}}\|_\infty) + \epsilon^2 (2 + \alpha \|\hat{\mathbf{G}}\|_\infty)^2$.

Robust LQG - suboptimality

Theorem

Let $\mathbf{K}_\star$ be the optimal LQG controller, and the corresponding closed-loop responses be $\mathbf{Y}_\star, \mathbf{U}_\star, \mathbf{W}_\star, \mathbf{Z}_\star$. Let $\hat{\mathbf{G}}$ be the plant estimation with error $\|\mathbf{\Delta}\|_\infty < \epsilon$, where $\mathbf{\Delta} = \mathbf{G}_\star - \hat{\mathbf{G}}$. Suppose that $\epsilon\|\mathbf{U}_\star\|_\infty < \frac{1}{5}$, and choose the constant hyper-parameter $\alpha \in \left[\frac{\sqrt{2}\|\mathbf{U}_\star\|_\infty}{1-\epsilon\|\mathbf{U}_\star\|_\infty}, \frac{1}{\epsilon}\right)$. We denote the optimal solution to (2) *as $\gamma_\star, \hat{\mathbf{Y}}_\star, \hat{\mathbf{U}}_\star, \hat{\mathbf{W}}_\star, \hat{\mathbf{Z}}_\star$. Then, when applying the resulting controller $\mathbf{K} = \hat{\mathbf{U}}_\star \hat{\mathbf{Y}}_\star^{-1}$ to the true plant $\mathbf{G}_\star$, the relative error in the LQG cost is upper bounded by*

$$\frac{J(\mathbf{G}_\star, \mathbf{K})^2 - J(\mathbf{G}_\star, \mathbf{K}_\star)^2}{J(\mathbf{G}_\star, \mathbf{K}_\star)^2} \leq 20\epsilon\|\mathbf{U}_\star\|_\infty + h(\epsilon, \alpha) + g(\epsilon, \|\mathbf{U}_\star\|_\infty), \qquad (3)$$

where

$$g(\epsilon, \|\mathbf{U}_\star\|_\infty) = \epsilon\|\mathbf{G}_\star\|_\infty(2 + \|\mathbf{U}_\star\|_\infty\|\mathbf{G}_\star\|_\infty) + \epsilon^2(2 + \|\mathbf{U}_\star\|_\infty\|\mathbf{G}_\star\|_\infty)^2. \qquad (4)$$

Optimality $\mathcal{O}(\epsilon^2)$ vs Robustness $\mathcal{O}(\epsilon)$: the price of obtaining a faster rate is that the certainty equivalent controller becomes less robust to uncertainty.

Outline

Motivation and problem formulation

Stability and convexity in frequency-domain

Robust formulation of LQR and its suboptimality

Robust formulation of LQG and its suboptimality

Summary

Summary

	$\min_{\mathbf{K}} \sup_{\|\Delta_A\|,\|\Delta_B\|<\epsilon} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}(x_t^{\mathsf{T}}Qx_t + u_t^{\mathsf{T}}Ru_t)\right]$ subject to $x_{t+1} = (\hat{A}+\Delta A)x_t + (\hat{B}+\Delta B)u_t + v_t$, $\mathbf{u} = \mathbf{K}\mathbf{x}$	$\min_{\mathbf{K}} \sup_{\|\boldsymbol{\Delta}\|_\infty<\epsilon} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}(y_t^{\mathsf{T}}Qy_t + u_t^{\mathsf{T}}Ru_t)\right]$ subject to $\mathbf{y} = (\hat{\mathbf{G}}+\boldsymbol{\Delta})\mathbf{u}+\mathbf{v}$, $\mathbf{u} = \mathbf{K}\mathbf{y}+\mathbf{w}$,
Sys ID methods	❖ Least squares $\|\hat{A}-A_\star\| \leq \epsilon_A, \|\hat{B}-B_\star\| \leq \epsilon_B,$	❖ Least squares $\|\boldsymbol{\Delta}\|_\infty := \|\mathbf{G}_\star - \hat{\mathbf{G}}\|_\infty < \epsilon$
Synthesis Technique	❖ Frequency domain ❖ System-level synthesis, SLS (Wang et al., 2019) ❖ Taylor expansion	❖ Frequency domain ❖ Input-output parameterization, IOP, (Furieri et al., 2019) ❖ Taylor expansion
Sample Complexity	❖ both stable and unstable systems $\frac{J(\hat{K})-J_\star}{J_\star} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$	❖ Only for open-loop stable system $\frac{J(\hat{\mathbf{K}})-J_\star}{J_\star} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$
References	✓ Dean et al., 2020; Berberich et al., 2020; Boczar et al., 2018; Tsiamis et al., 2020; Umenberger et al., 2019; and many others	• Zheng, Furieri, Kamgarpour, & Li, (2021, May). link

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Thank you for your attention!

Q & A