Sum-of-squares Chordal Decomposition of Polynomial Matrix Inequalities

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Outline

Matrix decomposition and chordal graphs

Sum-of-squares chordal decompositions

Applications to robust semidefinite optimization

Conclusions

Matrix decomposition and chordal graphs

Matrix decomposition:

- A simple example

$$A = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0}$$

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- This is true for any PSD matrix with such pattern, *i.e.*, sparse cone decomposition

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where $*$ denotes a real scalar number (or block matrix).

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where $*$ denotes a real scalar number (or block matrix).

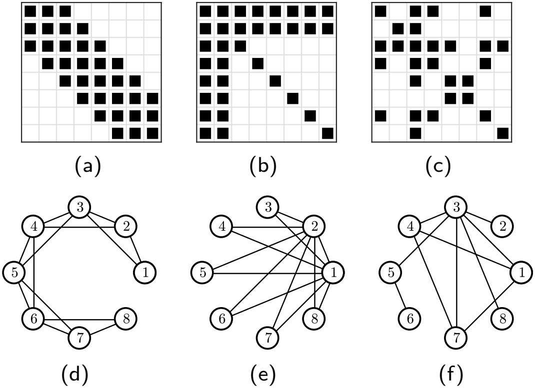
Benefits:

- Reduce computational complexity, and thus improve efficiency! $(3 \times 3 \rightarrow 2 \times 2)$

Matrix decomposition and chordal graphs

Matrix decomposition:

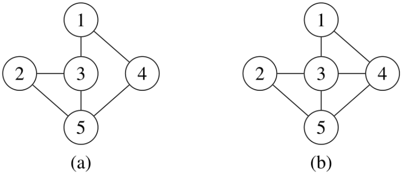
- Many other patterns admit similar decompositions, *e.g.*



- They can be commonly characterized by **chordal graphs**.

Clique decomposition

Chordal graphs: An undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *chordal* if every cycle of length greater than three has a chord.



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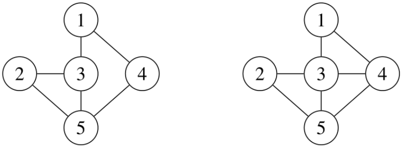

(a) (b)

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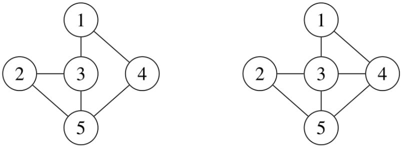

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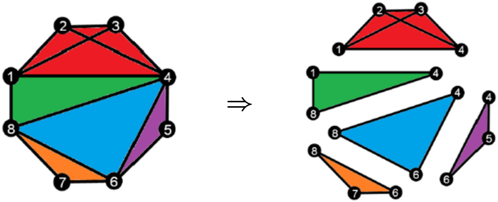

Sparse matrix decomposition

- **Sparse positive semidefinite (PSD) matrices**

$$\begin{aligned} \mathbb{S}^n(\mathcal{E},0) &= \{X \in \mathbb{S}^n \mid X_{ij} = X_{ji} = 0, \forall (i,j) \notin \mathcal{E}\}, \\ \mathbb{S}^n_+(\mathcal{E},0) &= \{X \in \mathbb{S}^n(\mathcal{E},0) \mid X \succeq 0\}. \end{aligned}$$

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- **Clique decomposition for PSD matrices** (Agler, Helton, McCullough, & Rodman, 1988; Griewank and Toint, 1984)

Theorem

Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be a chordal graph with maximal cliques $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. Then,

$$Z \in \mathbb{S}^n_+(\mathcal{E},0) \Leftrightarrow Z = \sum_{k=1}^{p} E_{\mathcal{C}_k}^{\mathsf{T}} Z_k E_{\mathcal{C}_k}, \; Z_k \in \mathbb{S}_+^{|\mathcal{C}_k|}$$

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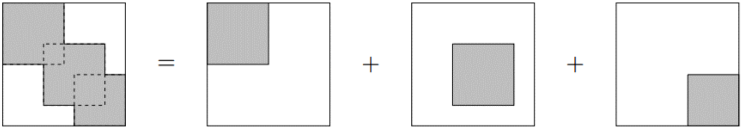

A growing number of applications

Control, machine learning, relaxation of QCQP, fluid dynamics, and beyond

Area	Topic	References
Control	Linear system analysis	Andersen et al. (2014b); Deroo et al. (2015); Mason & Papachristodoulou (2014); Pakazad et al. (2017b); Zheng et al. (2018c)
	Decentralized control	Deroo et al. (2014); Heinke et al. (2020); Zheng et al. (2020); Zheng et al. (2018d)
	Nonlinear system analysis	Schlosser & Korda (2020); Tacchi et al. (2019a); Zheng et al. (2019a); Mason (2015, Chapter 5)
	Model predictive control	Ahmadi et al. (2019); Hansson & Pakazad (2018)
Machine learning	Verification of neural networks	Batten et al. (2021); Dvijotham et al. (2020); Newton & Papachristodoulou (2021); Zhang (2020)
	Lipschitz constant estimation	Chen et al. (2020b); Latorre et al. (2020)
	Training of support vector machine	Andersen & Vandenberghe (2010)
	Geometric perception & coarsening	Chen et al. (2020a); Liu et al. (2019); Yang & Carlone (2020)
	Covariance selection	Dahl et al. (2008); Zhang et al. (2018)
	Subspace clustering	Miller et al. (2019a)
Relaxation of QCQP and POPs	Sensor network locations	Jing et al. (2019); Kim et al. (2009); Nie (2009)
	Max-Cut problem	Andersen et al. (2010a); Garstka et al. (2019); Zheng et al. (2020)
	Optimal power flow (OPF)	Andersen et al. (2014a); Dall'Anese et al. (2013); Jabr (2011); Jiang (2017); Molzahn & Hiskens (2014); Molzahn et al. (2013)
	State estimation in power systems	Weng et al. (2013); Zhang et al. (2017); Zhu & Giannakis (2014)
Others	Fluid dynamics	Arslan et al. (2021); Fantuzzi et al. (2018)
	Partial differential equations	Mevissen (2010); Mevissen et al. (2008, 2011, 2009)
	Robust quadratic optimization	Andersen et al. (2010b)
	Binary signal recovery	Fosson & Abuabiah (2019)
	Solving polynomial systems	Cifuentes & Parrilo (2016, 2017); Li et al. (2021); Mou et al. (2021); Tacchi et al. (2019b)
	Other problems	Baltean-Lugojan et al. (2019); Jeyakumar et al. (2016); Madani et al. (2017b); Pakazad et al. (2017a); Yang & Deng (2020)

- Zheng, Fantuzzi, & Papachristodoulou, (2021). Chordal and factor-width decompositions for scalable semidefinite and polynomial optimization. Annual Reviews in Control, 52, 243-279.

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Positive (semi)-definite polynomial matrices

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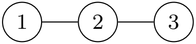
- **How about positive (semi)-definite polynomial matrices?**



$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & 0 \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ 0 & p_{32}(x) & p_{33}(x) \end{bmatrix} \succeq 0, \quad \forall x \in \mathcal{K}$$

$$\mathcal{K} = \mathbb{R}^n, \text{or}, \mathcal{K} = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

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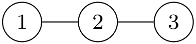

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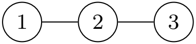

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- **Point-wise:** the decomposition still holds, but can it be represented by polynomials or even better, by SOS matrices?

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Naive extension does not work

Negative result

There exists a polynomial matrix $P(x)$ with chordal sparsity $\mathcal{G}$ that is strictly positive definite for all $x \in \mathbb{R}^n$, but cannot be decomposed with positive semidefinite polynomial matrices $S_k(x)$.

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$$P(x) = \begin{bmatrix} k+1+x^2 & x+x^2 & 0 \\ x+x^2 & k+2x^2 & x-x^2 \\ 0 & x-x^2 & k+1+x^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ x & x \\ 1 & -x \end{bmatrix} \begin{bmatrix} x & x & 1 \\ 1 & x & -x \end{bmatrix} + kI_3$$

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- It is not difficult to show that

$$P(x) = \underbrace{\begin{bmatrix} a(x) & b(x) & 0 \\ b(x) & c(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & d(x) & e(x) \\ 0 & e(x) & f(x) \end{bmatrix}}_{\succeq 0},$$

fails to exist when $0 \leq k < 2$.

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- $P(x)$ is strictly positive definite if $0 < k < 2$.

Sum-of-squares (SOS) matrices

- Consider a symmetric matrix-valued polynomial

$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \dots & p_{1r}(x) \\ p_{21}(x) & p_{22}(x) & \dots & p_{2r}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1}(x) & p_{r2}(x) & \dots & p_{rr}(x) \end{bmatrix} \succeq 0, \forall x \in \mathbb{R}^n.$$

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- **SOS representation:** We call $P(x)$ is an SOS matrix if

$$p(x, y) = y^{\mathsf{T}} P(x) y \text{ is SOS in } [x; y]$$

A polynomial $q(x)$ is SOS if it can be written as $q(x) = \sum_{i=1}^{m} f_i(x)^2$.

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- **SDP characterization (Parrilo *et al.*):** $P(x)$ is an SOS matrix if and only if there exists $Q \succeq 0$, such that

$$P(x) = (I_r \otimes v_d(x))^{\mathsf{T}} Q (I_r \otimes v_d(x)).$$

  where Q is called the Gram matrix, $v_d(x)$ is the standard monomial basis.

Hilbert–Artin theorem

Sparse matrix version of the Hilbert–Artin theorem

Let $P(x)$ be an $m \times m$ positive semidefinite polynomial matrix whose sparsity graph is chordal and has maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_t$. There exist an SOS polynomial $\sigma(x)$ and SOS matrices $S_k(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$\sigma(x)P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

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- **Example:** $\sigma(x) = 1 + k + x^2$ suffices for the previous example

$$P(x) = \begin{bmatrix} k+1+x^2 & x+x^2 & 0 \\ x+x^2 & \dfrac{(1+x)^2x^2}{1+k+x^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{k^2+k+3kx^2+(1-x)^2x^2}{1+k+x^2} & x-x^2 \\ 0 & x-x^2 & k+1+x^2 \end{bmatrix}$$

- PSD polynomial matrices are equivalent to SOS matrices when $n = 1$.

Reznick's Positivstellensatz

Sparse matrix version of Reznick's Positivstellensatz

Let $P(x)$ be an $m \times m$ homogeneous polynomial matrix whose sparsity graph is chordal and has maximal cliques $\mathcal{C}_1, \dots, \mathcal{C}_t$. If P is strictly positive definite on $\mathbb{R}^n \setminus \{0\}$, there exist an integer $\nu \geq 0$ and homogeneous SOS matrices $S_k(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$\|x\|^{2\nu} P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

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- **De-homogenization**: If P is strictly positive definite on $\mathbb{R}^n$ and its highest-degree homogeneous part $\sum_{|\alpha|=2d} P_\alpha x^\alpha$ is strictly positive definite on $\mathbb{R}^n \setminus \{0\}$, then, we have

$$(1 + \|x\|^2)^{\nu} P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

where $\nu \geq 0$ is an integer and $S_k(x)$ are SOS matrices of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$.

Reznick's Positivstellensatz

- **Non-trivial example:** Let $q(x) = x_1^2 x_2^4 + x_1^4 x_2^2 - 3x_1^2 x_2^2 + 1$ be the Motzkin polynomial, and

$$P(x) = \begin{bmatrix} 0.01(1 + x_1^6 + x_2^6) + q(x) & -0.01x_1 & 0 \\ -0.01x_1 & x_1^6 + x_2^6 + 1 & -x_2 \\ 0 & -x_2 & x_1^6 + x_2^6 + 1 \end{bmatrix}.$$

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- $P(x)$ is is strictly positive definite on $\mathbb{R}^2$, but is not SOS ($\varepsilon(1 + x_1^6 + x_2^6)$ $+q(x)$ is not SOS unless $\varepsilon \gtrsim 0.01006$ [Laurent 2009, Example 6.25]).

Reznick's Positivstellensatz

- **Non-trivial example:** Let $q(x) = x_1^2 x_2^4 + x_1^4 x_2^2 - 3x_1^2 x_2^2 + 1$ be the Motzkin polynomial, and

$$P(x) = \begin{bmatrix} 0.01(1 + x_1^6 + x_2^6) + q(x) & -0.01x_1 & 0 \\ -0.01x_1 & x_1^6 + x_2^6 + 1 & -x_2 \\ 0 & -x_2 & x_1^6 + x_2^6 + 1 \end{bmatrix}.$$

- $P(x)$ is is strictly positive definite on $\mathbb{R}^2$, but is not SOS ($\varepsilon(1 + x_1^6 + x_2^6)$ $+q(x)$ is not SOS unless $\varepsilon \gtrsim 0.01006$ [Laurent 2009, Example 6.25]).
- Our theorem guarantees the following decomposition exists

$$(1 + \|x\|^2)^\nu P(x) = E_{\mathcal{C}_1}^\mathsf{T} S_1(x) E_{\mathcal{C}_1} + E_{\mathcal{C}_2}^\mathsf{T} S_2(x) E_{\mathcal{C}_2}.$$

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- It suffices to use $\nu = 1$ and SOS matrices

$$S_1(x) = \begin{bmatrix} (1 + \|x\|^2)q(x) & 0 \\ 0 & 0 \end{bmatrix} + \frac{1 + \|x\|^2}{100} \begin{bmatrix} 1 + x_1^6 + x_2^6 & -x_1 \\ -x_1 & 100x_1^2 \end{bmatrix},$$

$$S_2(x) = (1 + \|x\|^2) \begin{bmatrix} 1 - x_1^2 + x_1^6 + x_2^6 & -x_2 \\ -x_2 & 1 + x_1^6 + x_2^6 \end{bmatrix}.$$

Putinar's Positivstellensatz

Consider $P(x) \succ 0, \forall x \in \mathcal{K}$ with $\mathcal{K} = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$, and

$$\sigma_0(x) + g_1(x)\sigma_1(x) + \cdots + g_q(x)\sigma_q(x) = r^2 - \|x\|^2.$$

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Sparse matrix version of Putinar's Positivstellensatz

Let $P(x)$ be a polynomial matrix whose sparsity graph is chordal and has maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_t$. If P is strictly positive definite on $\mathcal{K}$, there exist SOS matrices $S_{j,k}(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} \left(S_{0,k}(x) + \sum_{j=1}^{q} g_j(x) S_{j,k}(x) \right) E_{\mathcal{C}_k}.$$

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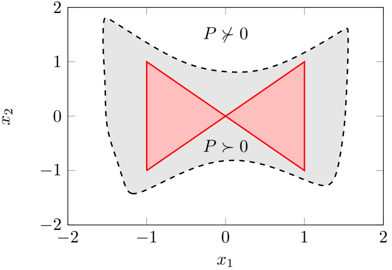
- **Example:** Let $\mathcal{K} = \{x \in \mathbb{R}^2 : g_1(x) := 1 - x_1^2 \geq 0, g_2(x) := x_1^2 - x_2^2 \geq 0\}$, and

$$P(x) := \begin{bmatrix} 1 + 2x_1^2 - x_1^4 & x_1 + x_1x_2 - x_1^3 & 0 \\ x_1 + x_1x_2 - x_1^3 & 3 + 4x_1^2 - 3x_2^2 & 2x_1^2x_2 - x_1x_2 - 2x_2^3 \\ 0 & 2x_1^2x_2 - x_1x_2 - 2x_2^3 & 1 + x_2^2 + x_1^2x_2^2 - x_2^4 \end{bmatrix}$$

Putinar's Positivstellensatz



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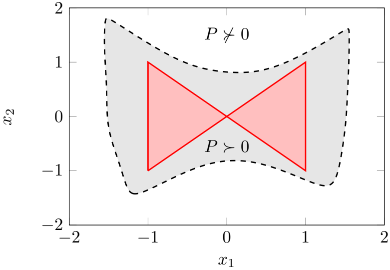

- It guarantees the decomposition below holds for SOS matrices $S_{i,j}(x)$

$$P(x) = \sum_{k=1}^{2} E_{\mathcal{C}_k}^{\mathsf{T}} \left[S_{0,k}(x) + g_1(x) S_{1,k}(x) + g_2(x) S_{2,k}(x) \right] E_{\mathcal{C}_k}$$

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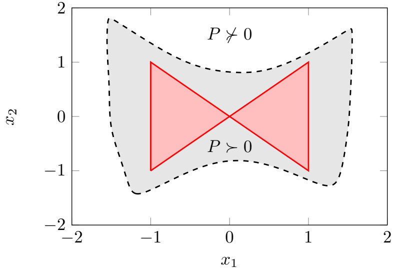

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- Possible choices are

$$S_{0,1}(x) = I_2 + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \qquad S_{1,1}(x) = \begin{bmatrix} x_1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \end{bmatrix}$$

$$S_{0,2}(x) = I_2 + \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \begin{bmatrix} x_1 & -x_2 \end{bmatrix} \qquad S_{2,2}(x) = \begin{bmatrix} 2 \\ x_2 \end{bmatrix} \begin{bmatrix} 2 & x_2 \end{bmatrix}.$$

Outline

Robust semidefinite optimization

Consider a robust SDP program

$$B^* := \inf_{\lambda \in \mathbb{R}^\ell} \quad b^\mathsf{T} \lambda$$

$$\text{subject to} \quad P(x, \lambda) := P_0(x) - \sum_{i=1}^{\ell} P_i(x)\lambda_i \succeq 0 \quad \forall x \in \mathcal{K},$$

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Convergence guarantees

- $\mathcal{K}$ is compact and satisfies the Archimedean condition, under some technical conditions, we fix $\sigma(x) = 1$ and $B^*_{d,0} \to B^*$ from above as $d \to \infty$.

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- $\mathcal{K} \equiv \mathbb{R}^n$: under some technical conditions, we fix $\sigma(x) = 1 + \|x\|^2$ and $B^*_{d,\nu} \to B^*$ from above as $\nu \to \infty$.

Numerical Experiments

Define a set

$$\mathcal{F}_\omega = \{\lambda \in \mathbb{R}^2 : \; P_\omega(x,\lambda) \succeq 0 \quad \forall x \in \mathbb{R}^3\}.$$

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- Define two hierarchies of subsets of $\mathcal{F}_\omega$, indexed by an integer ν, as

$$\mathcal{D}_{\omega,\nu} := \left\{\lambda \in \mathbb{R}^2 : \; \|x\|^{2\nu} P_\omega(x,\lambda) \text{ is SOS}\right\},$$

$$\mathcal{S}_{\omega,\nu} := \left\{\lambda \in \mathbb{R}^2 : \; \|x\|^{2\nu} P_\omega(x,\lambda) = \sum_{k=1}^{3\omega-1} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}, S_k(x) \text{ is SOS}\right\}.$$

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- We always have

$$\mathcal{S}_{\omega,\nu} \subseteq \mathcal{D}_{\omega,\nu} \subseteq \mathcal{F}_\omega$$

Numerical Experiments

We consider

$$\begin{aligned} B^* := \inf_{\lambda} \quad & \lambda_2 - 10\lambda_1 \\ \text{subject to} \quad & \lambda \in \mathcal{F}_\omega \end{aligned}$$

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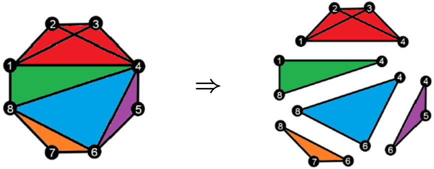
Table: Upper bounds $B_{d,\nu}$ on the optimal value B^* and time (seconds) by MOSEK

	Standard SOS						Sparse SOS					
	$\nu = 1$		$\nu = 2$		$\nu = 3$		$\nu = 2$		$\nu = 3$		$\nu = 4$	
ω	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$
5	12	-8.68	25	-9.36	69	-9.36	0.58	-8.97	0.72	-9.36	1.29	-9.36
10	407	-8.33	886	-9.09	2910	-9.09	1.65	-8.72	0.82	-9.09	2.08	-9.09
15	2090	-8.26	OOM	OOM	OOM	OOM	2.76	-8.68	1.13	-9.04	2.79	-9.04
20	OOM	OOM	OOM	OOM	OOM	OOM	3.24	-8.66	1.54	-9.02	4.70	-9.02
25	OOM	OOM	OOM	OOM	OOM	OOM	2.85	-8.66	1.94	-9.02	4.59	-9.02
30	OOM	OOM	OOM	OOM	OOM	OOM	2.38	-8.65	2.40	-9.01	5.50	-9.01
35	OOM	OOM	OOM	OOM	OOM	OOM	2.66	-8.65	3.25	-9.01	6.17	-9.01
40	OOM	OOM	OOM	OOM	OOM	OOM	3.07	-8.65	3.14	-9.01	8.48	-9.01

Outline

Summary

Clique decomposition:



- It offers a decomposition for sparse positive semidefinite matrices

$$Z \in \mathbb{S}^n_+(\mathcal{E}, 0) \Leftrightarrow Z = \sum_{k=1}^{p} E_{\mathcal{C}_k}^{\mathsf{T}} Z_k E_{\mathcal{C}_k}, \; Z_k \in \mathbb{S}_+^{|\mathcal{C}_k|}$$

- We present extensions to polynomial matrices: sparsity-exploiting versions of the **Hilbert-Artin**, **Reznick**, **Putinar** Positivstellensätze.
- Applications to robust semidefinite optimization with sparsity!

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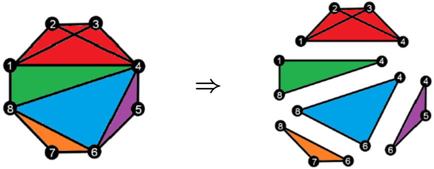

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- Applications to robust semidefinite optimization with sparsity!

Future work

- Polynomial matrix completion;
- Moment interpretations of the decomposition and completion results.

Thank you for your attention!

Q & A

- Zheng, Y., Fantuzzi, G., & Papachristodoulou, A. (2021). Chordal and factor-width decompositions for scalable semidefinite and polynomial optimization. Annual Reviews in Control, 52, 243-279.
- Zheng, Y., & Fantuzzi, G. (2023). Sum-of-squares chordal decomposition of polynomial matrix inequalities. Mathematical Programming, 197(1), 71-108.

Extra slides

Proof ideas: Hilbert–Artin theorem

Diagonalization with no fill-ins

If $P(x)$ is an $m \times m$ symmetric polynomial matrix with chordal sparsity graph, there exist an $m \times m$ permutation matrix T, an invertible $m \times m$ lower-triangular polynomial matrix $L(x)$, and polynomials $b(x)$, $d_1(x)$, $\ldots$, $d_m(x)$ such that

$$b^4(x)\, T P(x) T^\mathsf{T} = L(x) \mathsf{Diag}\left(d_1(x), \ldots, d_m(x)\right) L(x)^\mathsf{T}.$$

Moreover, L has no fill-in in the sense that $L + L^\mathsf{T}$ has the same sparsity as TPT^T.

Proof ideas: Hilbert–Artin theorem

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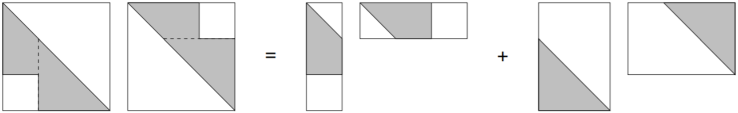

Figure: Decomposition follows by combining columns.

Figure from Prof. Lieven Vandenberghe's talk.

Proof ideas: Putinar's theorem

Scherer and Ho, 2006

Let $\mathcal{K}$ be a compact semialgebraic set that satisfies the Archimedean condition. If an $m \times m$ symmetric polynomial matrix $P(x)$ is strictly positive definite on $\mathcal{K}$, there exist $m \times m$ SOS matrices $S_0, \ldots, S_q$ such that

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- Weierstrass polynomial approximation theorem + the above version of Putinar's Positivstellensatz

$$P(x) = \begin{bmatrix} a(x) & b(x)^\mathsf{T} & 0 \\ b(x) & U(x) & V(x) \\ 0 & V(x) & W(x) \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} a(x) & b(x)^\mathsf{T} & 0 \\ b(x) & H(x) + 2\varepsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0, \forall x \in \mathcal{K}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & U(x) - H(x) - 2\varepsilon I & V(x) \\ 0 & V(x)^\mathsf{T} & W(x) \end{bmatrix}}_{\succeq 0, \forall x \in \mathcal{K}}.$$