Error bounds, PL condition, and quadratic growth for weakly convex functions, and linear convergences of proximal point methods

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Overview

- (sub)gradient-based methods and their variants are the workhorse algorithms for machine learning applications.
- Well understood for these methods:

  Smoothness + Strong convexity ⇒ Linear convergence

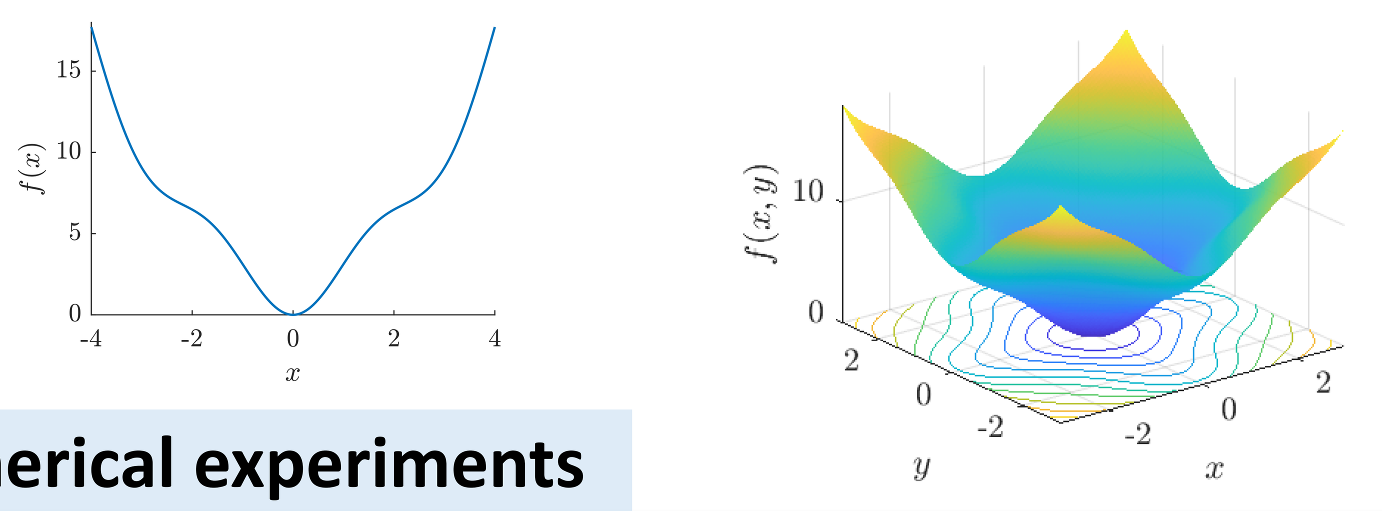
- However, **smoothness** and **strong convexity** are often not satisfied in practice
- Motivate the study of other weaker regularity conditions that also guarantee linear convergence, including RSI, EB, PL, and QG.
- In the class of **weakly convex** functions, common regularity conditions possess certain relationship/equivalence.
- Using the established equivalence in **convex optimization**, we provide
  - **Linear convergence** of the proximal point method (PPM).
  - **Simple** and **clean** proof.
- The linear convergence result extends to **weakly convex** functions.
- Proper control of the inexactness ⇒ Linear convergence of **inexact** PPM.

Preliminary

- Consider a ρ-weakly convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ ($f + \frac{\rho}{2}\|\cdot\|^2$ is convex).
- Define the Fréchet subdifferential

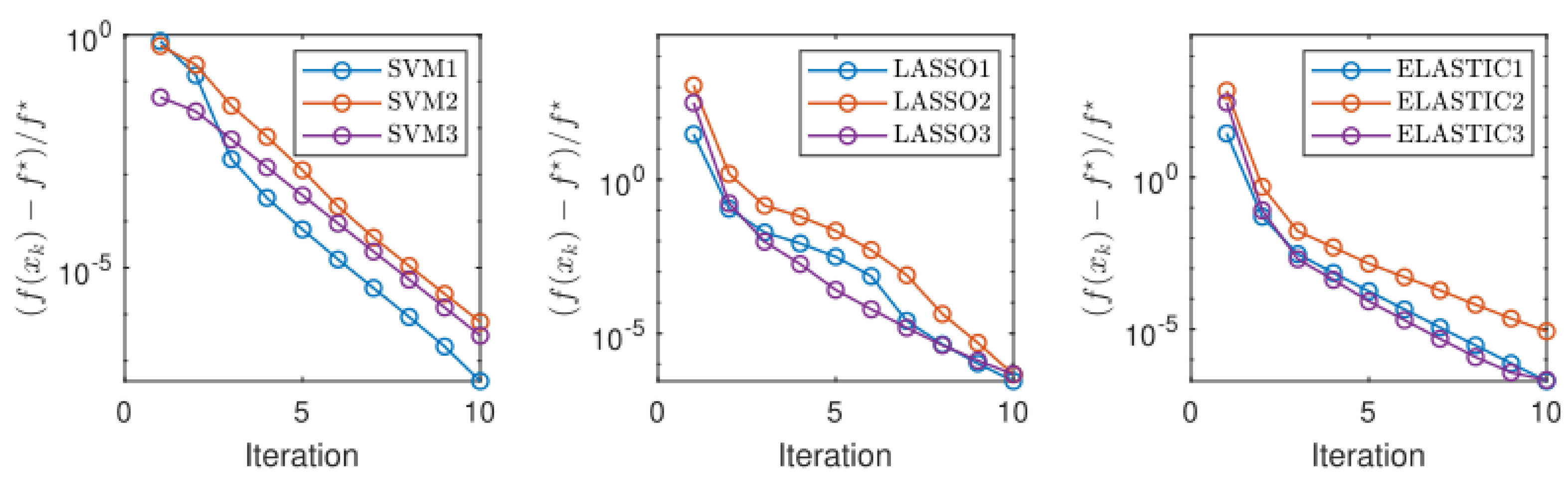
$$\hat{\partial} f(x) = \left\{ s \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{f(y) - f(x) - \langle s, y - x\rangle}{\|y - x\|} \geq 0 \right\}.$$

- Let $S = \operatorname{argmin}_x f(x)$ and $\nu > 0$ and consider the following regularity conditions
  - Local strongly convex **(SC)**: $\exists \mu_{\mathrm{s}} > 0$,
  $$f(x) + \langle g, y - x\rangle + \frac{\mu_{\mathrm{s}}}{2}\|y - x\|^2 \leq f(y),\ \forall x, y \in [f \leq f^\star + \nu], g \in \hat{\partial} f(x).$$
  - Restricted secant inequality **(RSI)**: $\exists \mu_{\mathrm{r}} > 0$,
  $$\mu_{\mathrm{r}} \cdot \operatorname{dist}^2(x, S) \leq \langle g, x - \hat{x}\rangle,\ \forall x \in [f \leq f^\star + \nu], g \in \hat{\partial} f(x), \hat{x} \in \Pi_S(x).$$
  - Polyak-Lojasiewicz **(PL)**: $\exists \mu_{\mathrm{p}} > 0$,
  $$2\mu_{\mathrm{p}} \cdot (f(x) - f^\star) \leq \operatorname{dist}^2(0, \hat{\partial} f(x)),\ \forall x \in [f \leq f^\star + \nu].$$
  - Error bound **(EB)**: $\exists \mu_{\mathrm{e}} > 0$,
  $$\operatorname{dist}(x, S) \leq \mu_{\mathrm{e}} \cdot \operatorname{dist}(0, \hat{\partial} f(x)),\ \forall x \in [f \leq f^\star + \nu].$$
  - Quadratic Growth **(QG)**: $\exists \mu_{\mathrm{q}} > 0$,
  $$\frac{\mu_{\mathrm{q}}}{2} \cdot \operatorname{dist}^2(x, S) \leq f(x) - f^\star,\ \forall x \in [f \leq f^\star + \nu].$$
- **SC**, **RSI**, **EB**, and **PL** imply that all stationary points are global optimal.
- The following two **nonconvex** functions satisfy **RSI**, **EB**, and **PL**



Numerical experiments

- Three machine learning applications



Reference

❒ Feng-Yi Liao, Lijun Ding, and Yang Zheng. Error bounds, PL condition, and quadratic growth for weakly convex functions, and linear convergences of proximal point methods. arXiv preprint arXiv:2307.07651, 2023.

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Main Result 1: Relationship/Equivalence

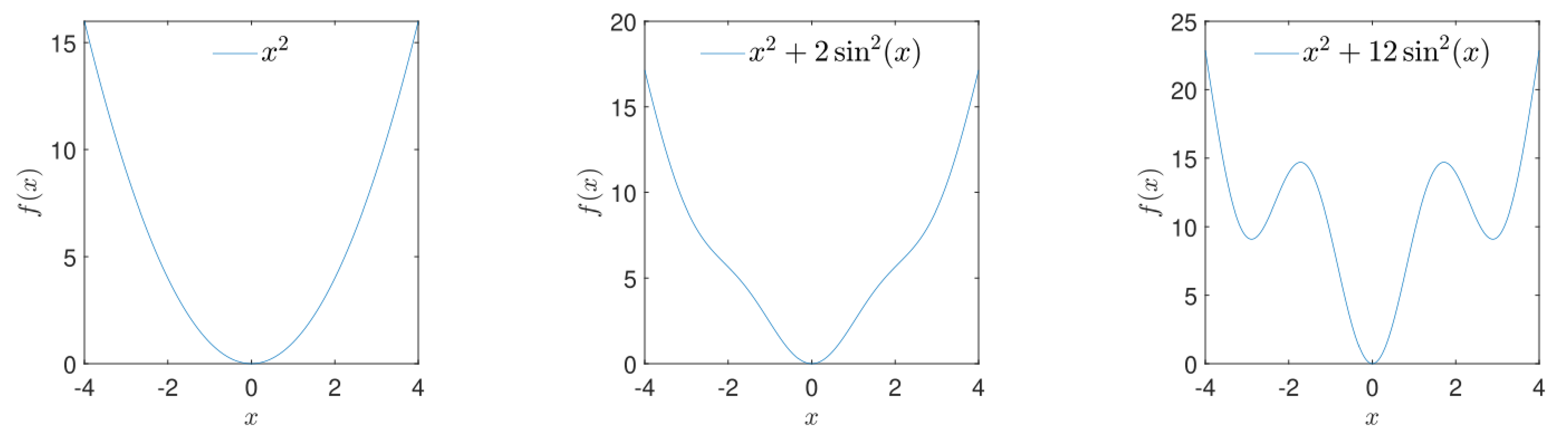
- Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper closed ρ-weakly convex function. Then

$$\textbf{SC} \to \textbf{RSI} \to \textbf{EB} \equiv \textbf{PL} \to \textbf{QG}.$$

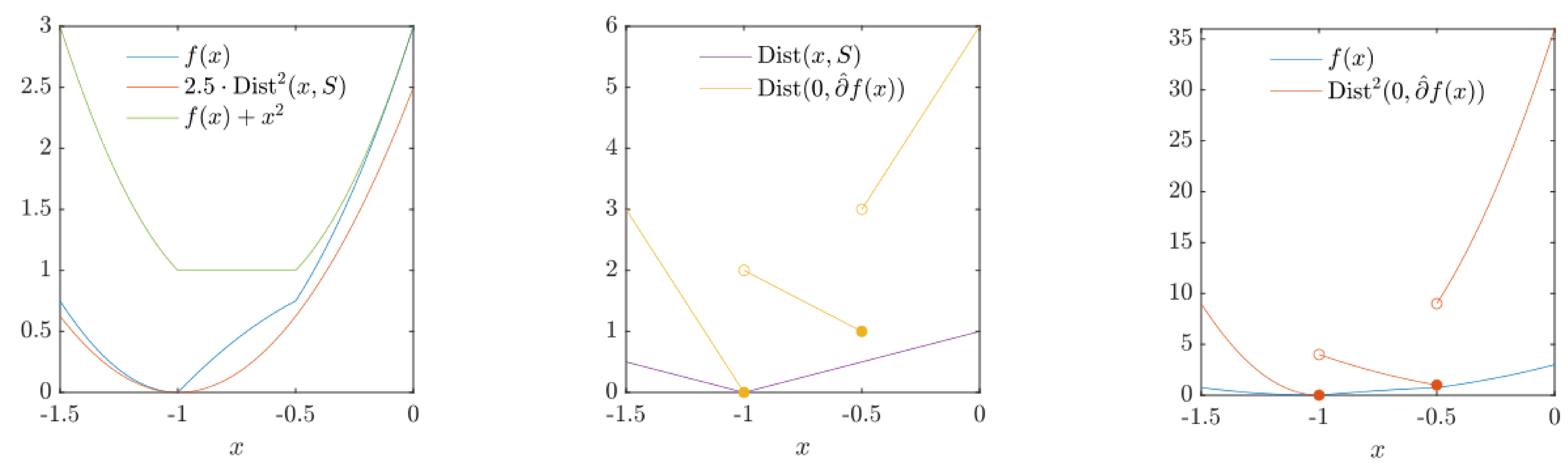
- Furthermore, if the coefficient of **QG** satisfies $\mu_{\mathrm{q}} > \rho$ (including the function f is convex), then the following equivalence holds

$$\textbf{RSI} \equiv \textbf{EB} \equiv \textbf{PL} \equiv \textbf{QG}.$$

- **QG** is the most genral condition.
- **Example 1:** $f(x) = x^2$. f is convex and all properties hold!
- **Example 2:** $f(x) = x^2 + 2\sin^2(x)$. All properties hold but f is not convex!
- **Example 3:** $f(x) = x^2 + 12\sin^2(x)$. All properties fail except **QG**.



- **Example 4:** A **nonsmooth** and **nonconvex** function satisfying $\mu_{\mathrm{q}} > \rho$.



Main result 2: Linear convergence of PPM

- Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \ f(x)$$

where $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is a proper closed **convex** function.

- PPM follows the update formula

$$x_{k+1} = \operatorname*{argmin}_{x \in \mathbb{R}^n} \ f(x) + \frac{1}{2c_k}\|x - x_k\|^2,\ k = 0, 1, 2, \ldots. \tag{1}$$

- Suppose f satisfies **PL** (or **EB**, **RSI**, **QG**) globally. Then, the iterates of PPM with a positive sequence $\{c_k\}_{k\geq 0}$ satisfies

$$f(x_{k+1}) - f^\star \leq \omega_k \cdot (f(x_k) - f^\star), \quad \omega_k < 1,$$
$$\operatorname{dist}(x_{k+1}, S) \leq \theta_k \cdot \operatorname{dist}(x_k, S), \quad \theta_k < 1.$$

- Suppose f is **ρ-weakly convex** and satisfy **QG** with $\mu_{\mathrm{q}} > \rho$ globally. Then, the iterates of PPM with a positive sequence $\{c_k\}_{k\geq 0}$ satisfying $\frac{1}{c_k} > \rho$ enjoy the linear convergence as the convex case.
- Simple proofs:

$$f(x_k) - f(x_{k+1}) \geq \|x_{k+1} - x_k\|^2/(2c_k) \overset{\textbf{O.C.}}{\geq} \frac{c_k}{2}\operatorname{dist}^2(0, \partial f(x_{k+1}))$$
$$\overset{\textbf{PL}}{\geq} c_k\mu_{\mathrm{p}}(f(x_{k+1}) - f^\star)$$
$$\Longrightarrow \frac{1}{1 + c_k\mu_{\mathrm{p}}}(f(x_k) - f^\star) \geq f(x_{k+1}) - f^\star.$$

- If (1) is solved **inexactly**, i.e.,

$$x_{k+1} \approx \operatorname*{argmin}_{x \in \mathbb{R}^n} \ f(x) + \frac{1}{2c_k}\|x - x_k\|^2,\ k = 0, 1, 2, \ldots,$$

with a proper control of inexactness and if f satisfy **QG**, then there exist a nonnegative $\theta < 1$ and a large $\bar{k} > 0$ such that

$$\forall k \geq \bar{k},\ \operatorname{dist}(x_{k+1}, S) \leq \theta_k \operatorname{dist}(x_k, S),\ \theta_k \leq \theta.$$

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