

# Nonconvex Optimization for Linear Quadratic Gaussian (LQG) Control

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**Control, Learning, and Verification Workshop @ NUS**

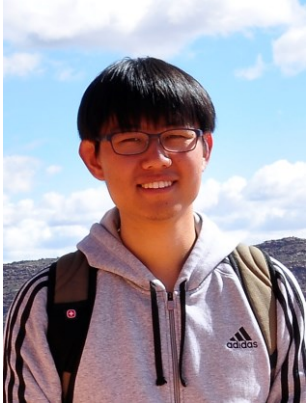
Dec 11, 2023

**UC San Diego**  
**JACOBS SCHOOL OF ENGINEERING**  
Electrical and Computer Engineering

Scalable Optimization and  
Control (SOC) Lab

<https://zhengy09.github.io/soclab.html>

# Acknowledgements



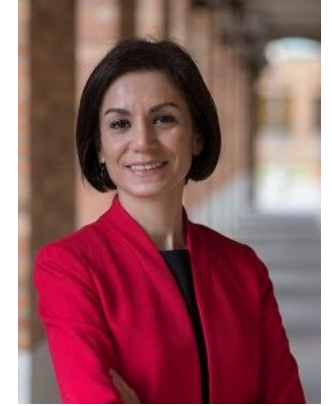
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1. Y. Tang\*, Y. Zheng\*, and N. Li, "Analysis of the optimization landscape of Linear Quadratic Gaussian (LQG) control," Mathematical Programming, 2023. Available: <https://arxiv.org/abs/2102.04393> \*Equal contribution
2. Y. Zheng\*, Y. Sun\*, M. Fazel, and N. Li. "Escaping High-order Saddles in Policy Optimization for Linear Quadratic Gaussian (LQG) Control." arXiv preprint, 2022 <https://arxiv.org/abs/2204.00912>. \*Equal contribution



Supported by  
NSF ECCS-2154650

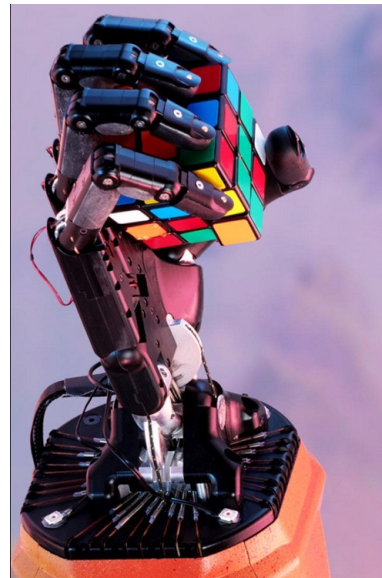
# Motivation

## □ Model-free methods and data-driven control

- Use direct policy updates
- Become very popular in both academia and practice, from game playing, robotics, and drones, etc.



DeepMind



OpenAI

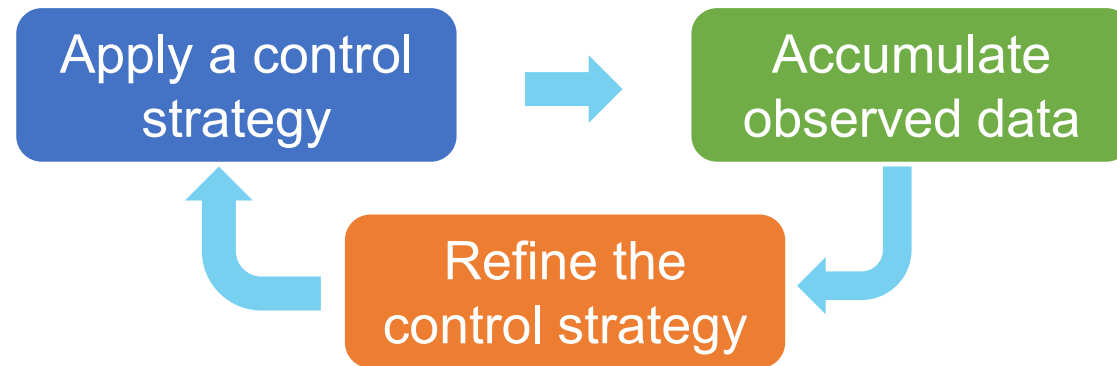


Applications

Duan et al. 2016; Silver et al., 2017; Dean et al., 2019; Tu and Recht, 2019;  
Mania et al., 2019; Fazel et al., 2018; Recht, 2019;

# Motivation

## □ Model-free methods and data-driven control



### Opportunities

- Directly search over a given policy class
- Directly optimize performance on the true system, bypassing the model estimation (not on an approximated model)

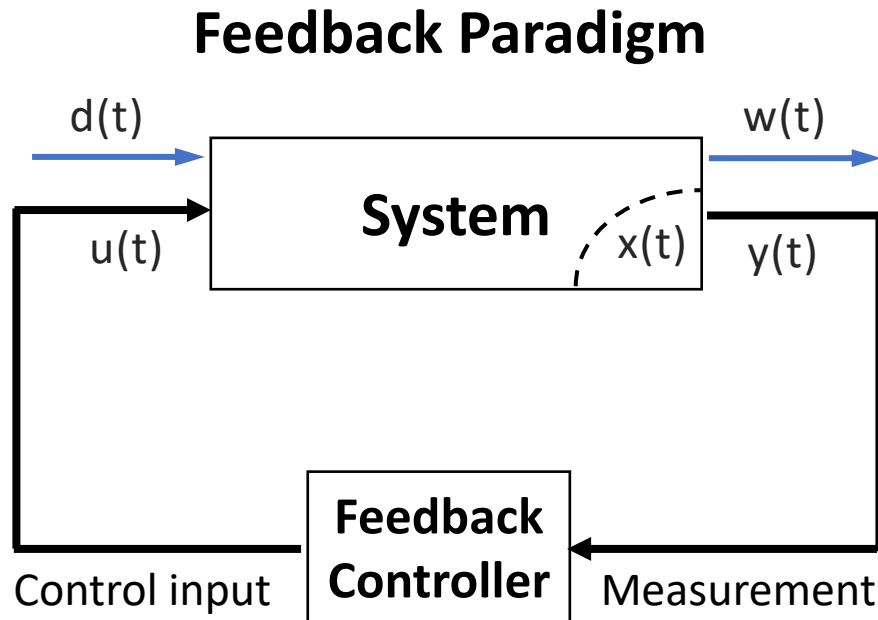
### Challenges

- Lack of non-asymptotic performance guarantees
  - Convergence
  - Suboptimality
  - Sample complexity, etc.

❖ Highly nontrivial even for **linear dynamical systems**

# Today's talk

## □ Optimal Control



**Control theory:** the principled use of feedback loops and algorithms to drive a dynamical system to its desired goal

## Linear Quadratic Optimal control

$$\min_{u_1, u_2, \dots,} \lim_{T \rightarrow \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T (x_t^\top Q x_t + u_t^\top R u_t) \right]$$

subject to  $x_{t+1} = A x_t + B u_t + w_t$   
 $y_t = C x_t + v_t$

- Many practical applications
- **Linear Quadratic Regulator (LQR)** when the state  $x_t$  is directly observable
- **Linear Quadratic Gaussian (LQG) control** when only partial output  $y_t$  is observed
- Extensive classical results (Dynamic programming, Separation principle, Riccati equations, etc.)

**Major challenge:** how to perform optimal control when the system is unknown?

# Direct policy optimization

## □ Controller parameterization

- Give a parameterization of control policies; say **neural networks?** ❌
- Control theory already tells us many structural properties
- **Linear feedback is sufficient for LQR**

$$u_t = Kx_t, \quad K \in \mathbb{R}^{m \times n}$$

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T (x_t^\top Q x_t + u_t^\top R u_t) \right] := J(K)$$

- Set of stabilizing controllers  $K \in \mathcal{K}$
- A fast-growing list of references

➤ Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others

## Policy optimization for LQR

$$\begin{aligned} \min_K \quad & J(K) \\ \text{s.t.} \quad & K \in \mathcal{K} \end{aligned}$$

## Direct policy iteration

$$K_{i+1} = K_i - \alpha_i \nabla J(K_i)$$

- ✓ Good optimization landscape properties (Fazel et al., 2018);
  - Connected feasible region
  - Unique stationary point
  - Gradient dominance
- ✓ Fast global convergence (linear)

# Challenges for partially observed LQG

## □ Results on model-free LQG control are much fewer

- LQG is more complicated than LQR
- Requires dynamical controllers
- Its non-convex landscape properties are much richer and more complicated than LQR

### Our focus: nonconvex optimization of LQG

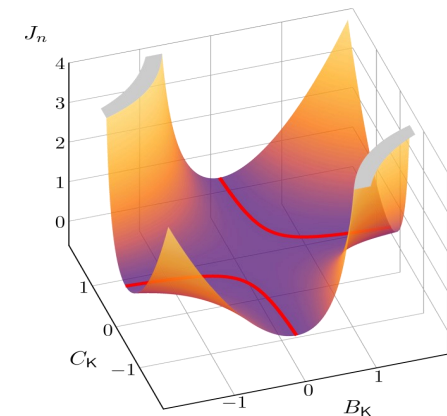
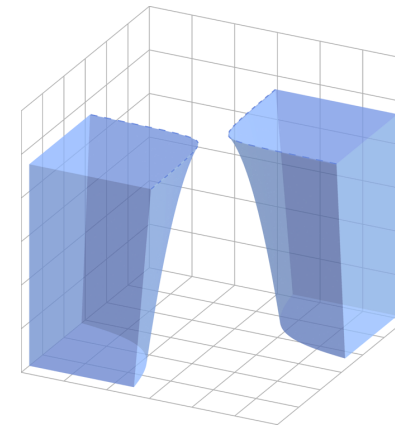
#### ■ Q1: Properties of the domain (set of stabilizing controllers)

- convexity, connectivity, open/closed?

#### ■ Q2: Properties of the accumulated cost

- convexity, differentiability, coercivity?
- set of stationary points/local minima/global minima?

#### ■ Q3: Escape saddle points via Perturbed Gradient Descent (PGD)



# Outline

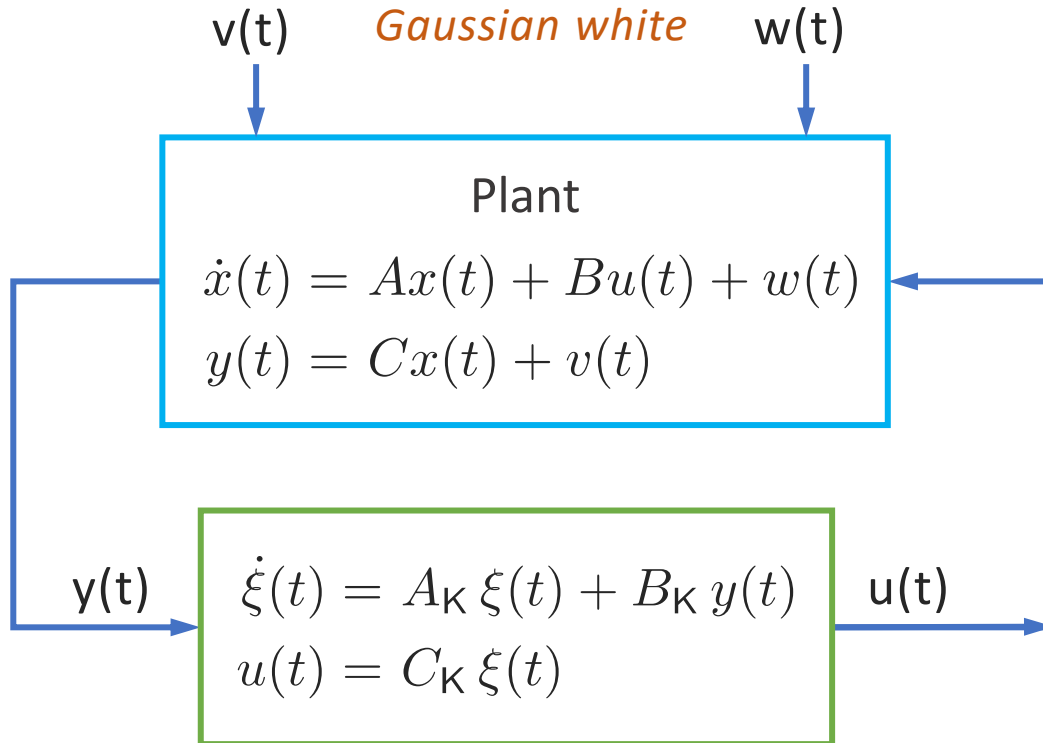
- ❑ **LQG problem Setup**
- ❑ **Connectivity of the Set of Stabilizing Controllers**
- ❑ **Structure of Stationary Points of the LQG cost**
- ❑ **Escaping saddle points via PGD**



# Outline

- LQG problem Setup**
- Connectivity of the Set of Stabilizing Controllers
- Structure of Stationary Points of the LQG cost
- Escaping saddle points via PGD

# LQG Problem Setup



dynamical controller

$$K = (A_K, B_K, C_K)$$

Standard Assumption	$(A, B), (A, W^{1/2})$	Controllable
	$(C, A), (Q^{1/2}, A)$	Observable

**Objective:** The LQG cost

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u) dt$$

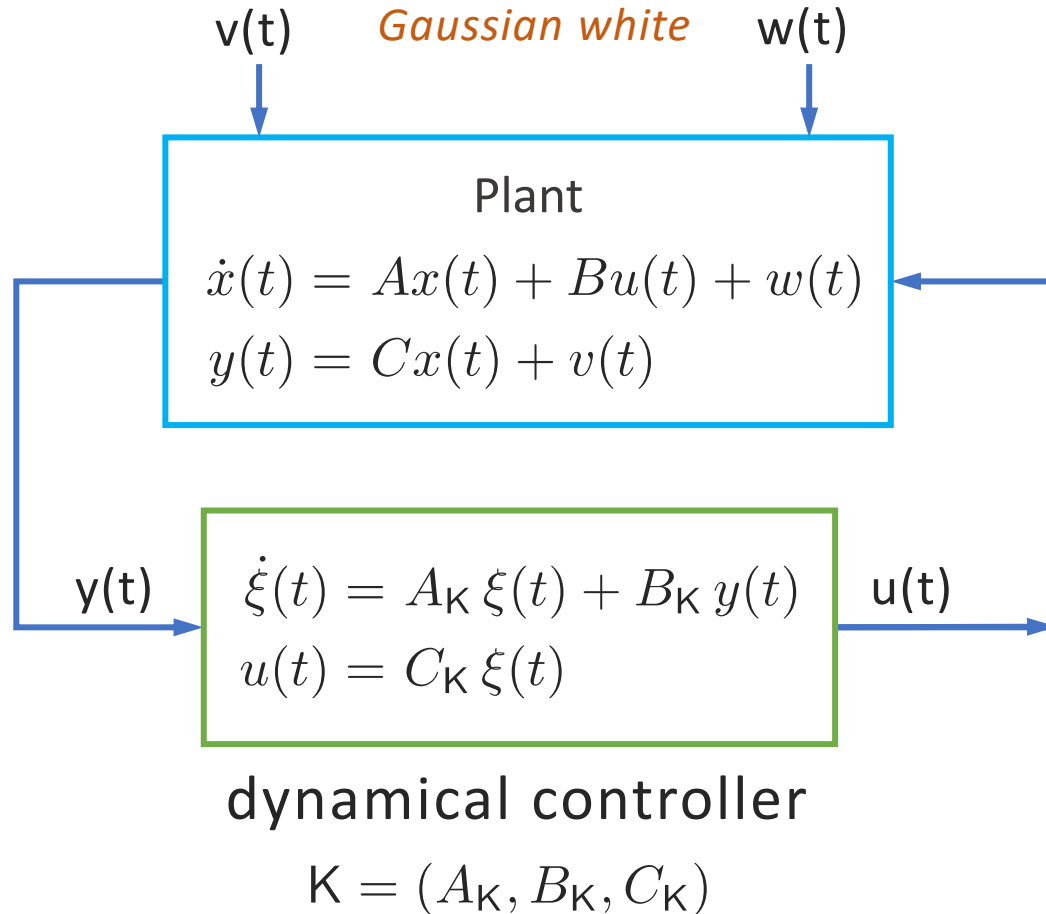
- $\xi(t)$  internal state of the controller
- $\dim \xi(t)$  order of the controller
- $\dim \xi(t) = \dim x(t)$  full-order
- $\dim \xi(t) < \dim x(t)$  reduced-order

## Minimal controller

The input-output behavior cannot be replicated by a lower order controller.

\*  $(A_K, B_K, C_K)$  controllable and observable

# Separation principle



Objective: The LQG cost

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u) dt$$

Solution: Kalman filter for state estimation  
+ LQR based on the estimated state

$$\dot{\xi} = (A - BK)\xi + L(y - C\xi),$$

$$u = -K\xi.$$

Two Riccati equations

➤ Kalman gain  $L = PC^\top V^{-1}$

$$AP + PA^\top - PC^\top V^{-1} CP + W = 0,$$

➤ Feedback gain  $K = R^{-1} B^\top S$

$$A^\top S + SA - SBR^{-1} B^\top S + Q = 0$$

Explicit dependence on the dynamics

# Policy Optimization formulation

## □ Closed-loop dynamics

$$\frac{d}{dt} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix},$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C_K \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}.$$

## □ Feasible region of the controller parameters

$$\mathcal{C}_{\text{full}} = \left\{ K \mid K = (A_K, B_K, C_K) \text{ is full order} \right. \\ \left. \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \text{ is Hurwitz stable} \right\}$$

## □ Cost function

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u) dt$$

$$J(K) = \text{tr} \left( \begin{bmatrix} Q & 0 \\ 0 & C_K^\top R C_K \end{bmatrix} X_K \right) = \text{tr} \left( \begin{bmatrix} W & 0 \\ 0 & B_K V B_K^\top \end{bmatrix} Y_K \right)$$

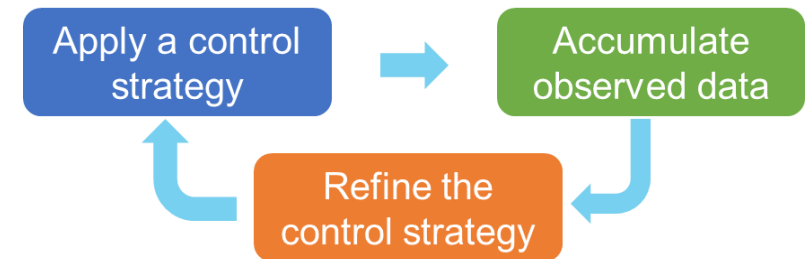
$X_K, Y_K$  Solution to Lyapunov equations

## Policy optimization for LQG

$$\min_K J(K)$$

$$\text{s.t. } K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$$

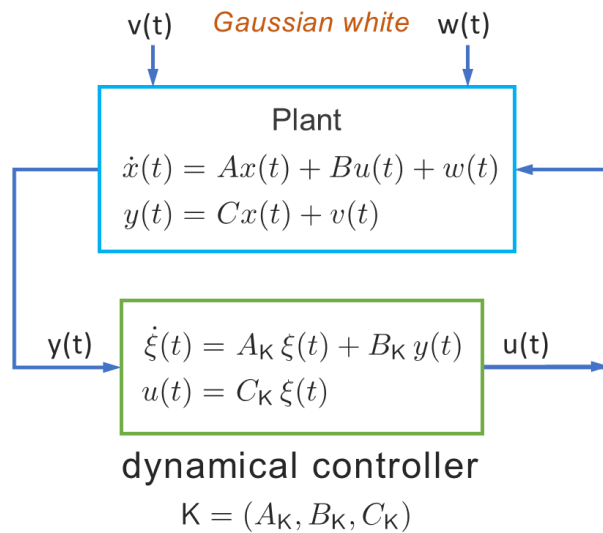
Direct policy iteration  $K_{i+1} = K_i - \alpha_i \nabla J(K_i)$



- ✓ Does it converge at all?
- ✓ Converge to which point?
- ✓ Convergence speed?

**Optimization  
Landscape  
Analysis**

# Main questions



## Policy optimization for LQG

$$\min_K J(K)$$

$$\text{s.t. } K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$$

## Non-convex Landscape Analysis

- **Q1: Connectivity of the feasible region  $\mathcal{C}_{\text{full}}$** 
  - Is it connected?
  - If not, how many connected components can it have?
- **Q2: Structure of stationary points of  $J(K)$** 
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?
- **Q3: How to escape high-order saddle points via PGD?**

# Outline

- LQG problem Setup
- **Connectivity of the Set of Stabilizing Controllers**
- Structure of Stationary Points of the LQG cost
- Escaping saddle points via PGD

# Connectivity of the feasible region

## □ Simple observation: non-convex and unbounded

**Lemma 1:** the set  $\mathcal{C}_{\text{full}}$  is non-empty, unbounded, and can be non-convex.

### Example

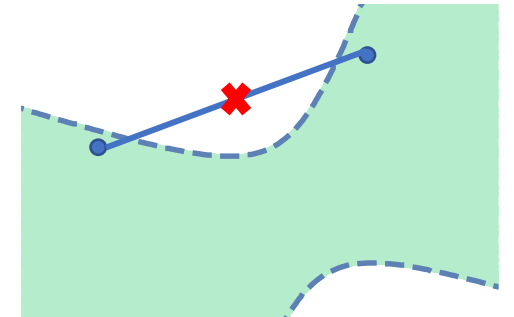
$$\dot{x}(t) = x(t) + u(t) + w(t)$$

$$y(t) = x(t) + v(t)$$

$$\mathcal{C}_{\text{full}} = \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid \begin{bmatrix} 1 & C_K \\ B_K & A_K \end{bmatrix} \text{ is stable} \right\}.$$

$$K^{(1)} = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}, \quad K^{(2)} = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix} \quad \text{Stabilize the plant, and thus belong to } \mathcal{C}_{\text{full}}$$

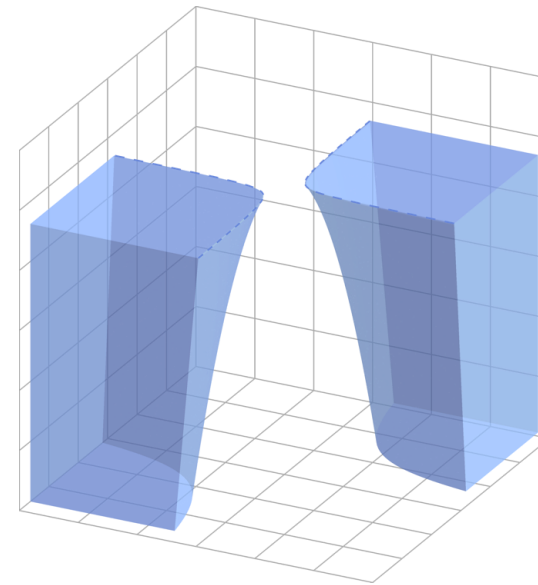
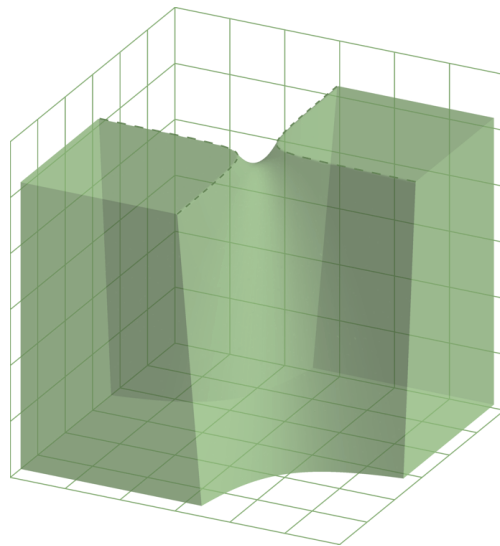
$$\hat{K} = \frac{1}{2} \left( K^{(1)} + K^{(2)} \right) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{Fails to stabilize the plant, and thus outside } \mathcal{C}_{\text{full}}$$



# Connectivity of the feasible region

## □ Main Result 1: dis-connectivity

**Theorem 1:** The set  $\mathcal{C}_{\text{full}}$  can be disconnected but has at most 2 connected components.



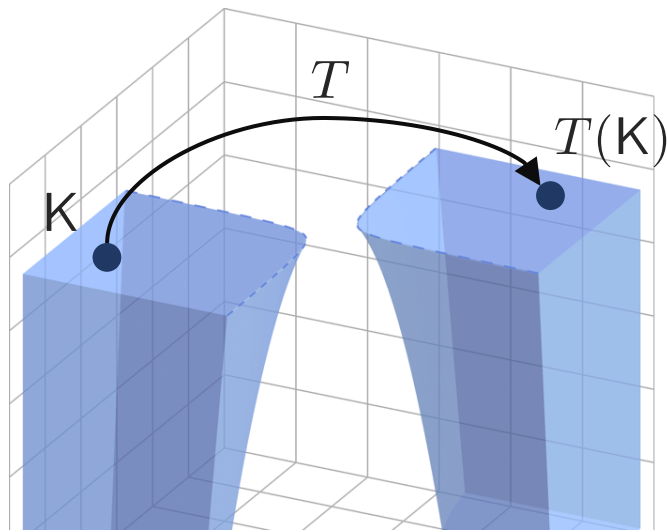
- ✓ Different from the connectivity of static stabilizing state-feedback controllers, which is always connected!
- ✓ Is this a negative result for gradient-based algorithms? → **No**



# Connectivity of the feasible region

## □ Main Result 2: dis-connectivity

**Theorem 2:** If  $\mathcal{C}_{\text{full}}$  has 2 connected components, then there is a smooth bijection  $T$  between the 2 connected components that has the same cost function value.



$$J(\mathbf{K}) = J(T(\mathbf{K}))$$

✓ In fact, the bijection  $T$  is defined by a similarity transformation (change of controller state coordinates)

$$\mathcal{I}_T(\mathbf{K}) := \begin{bmatrix} D_{\mathbf{K}} & C_{\mathbf{K}}T^{-1} \\ TB_{\mathbf{K}} & TA_{\mathbf{K}}T^{-1} \end{bmatrix}.$$

**Positive news:** For gradient-based local search methods, it makes no difference to search over either connected component.

# Connectivity of the feasible region

## □ Main Result 3: conditions for connectivity

**Theorem 3:** 1)  $\mathcal{C}_{\text{full}}$  is connected if there exists a reduced-order stabilizing controller.

2) The sufficient condition above becomes necessary if the plant is single-input or single-output.

**Corollary 1:** Given any open-loop stable plant, the set of stabilizing controllers  $\mathcal{C}_{\text{full}}$  is connected.

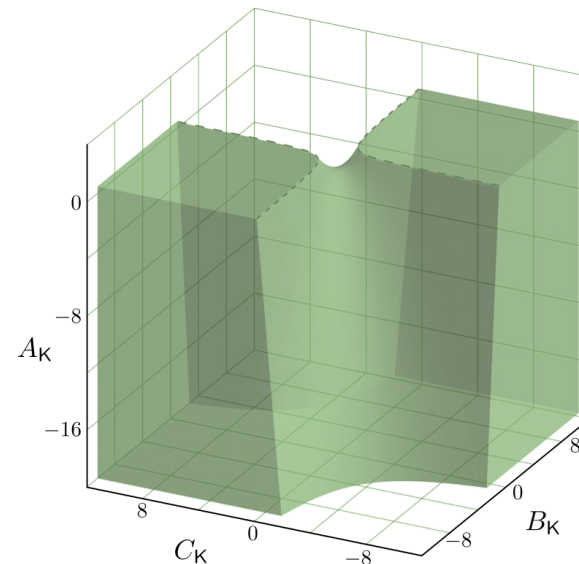
### Example: Open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$

$$y(t) = x(t) + v(t)$$

### Routh--Hurwitz stability criterion

$$\mathcal{C}_{\text{full}} = \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < 1, B_K C_K < -A_K \right\}.$$



# Connectivity of the feasible region

## □ Main Result 3: conditions for connectivity

### Example: Open-loop unstable system (SISO)

$$\dot{x}(t) = x(t) + u(t) + w(t)$$

$$y(t) = x(t) + v(t)$$

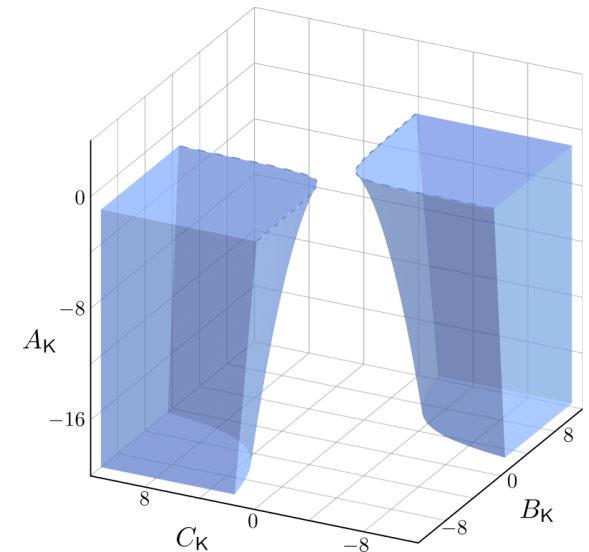
- **Routh--Hurwitz stability criterion**

$$\begin{aligned} \mathcal{C}_{\text{full}} &= \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \text{ is stable} \right\} \\ &= \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < -1, B_K C_K < A_K \right\}. \end{aligned}$$

- **Two path-connected components**

$$\begin{aligned} \mathcal{C}_1^+ &:= \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < -1, B_K C_K < A_K, B_K > 0 \right\}, \\ \mathcal{C}_1^- &:= \left\{ K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathbb{R}^{2 \times 2} \mid A_K < -1, B_K C_K < A_K, B_K < 0 \right\}. \end{aligned}$$

Disconnected feasible region



# Proof idea: Lifting via Change of Variables

## □ Change of variables in state-space domain: Lyapunov theory

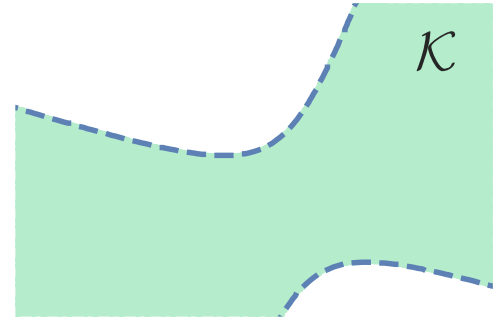
- Connectivity of the static stabilizing state feedback gains

$$\{K \in \mathbb{R}^{m \times n} \mid A - BK \text{ is stable}\}$$

$$\iff \{K \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, P(A - BK)^\top + (A - BK)P \prec 0\}$$

$$\iff \{K \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, PA^\top - L^\top B^\top + AP - BL \prec 0, L = KP\}$$

$$\iff \{K = LP^{-1} \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, PA^\top - L^\top B^\top + AP - BL \prec 0\}.$$



Open, connected,  
possibly nonconvex

- How about the set of stabilizing dynamical controllers

$$\begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \text{ is stable}$$

$$\iff \exists P \succ 0, P \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix}^\top + \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} P \prec 0,$$

Change of variables for  
output feedback control  
is highly non-trivial

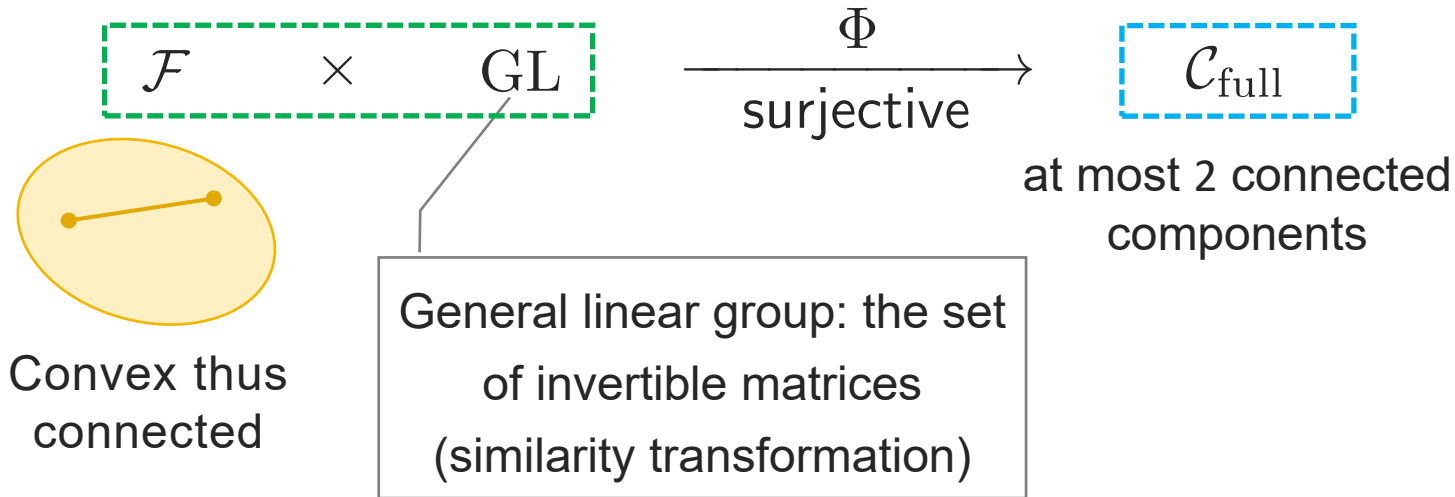
[Gahinet and Apkarian, 1994]  
[Scherer et al., IEEE TAC 1997]

# Proof idea: Lifting via Change of Variables

## □ Change of variables in state-space domain: Lyapunov theory

$$\Phi(Z) = \begin{bmatrix} \Phi_D(Z) & \Phi_C(Z) \\ \Phi_B(Z) & \Phi_A(Z) \end{bmatrix} := \begin{bmatrix} I & 0 \\ YB & \Xi \end{bmatrix}^{-1} \begin{bmatrix} G & H \\ F & M - YAX \end{bmatrix} \begin{bmatrix} I & CX \\ 0 & \Pi \end{bmatrix}^{-1} .$$

[Scherer et al., IEEE TAC 1997]  
[Gahinet and Apkarian, 1994]



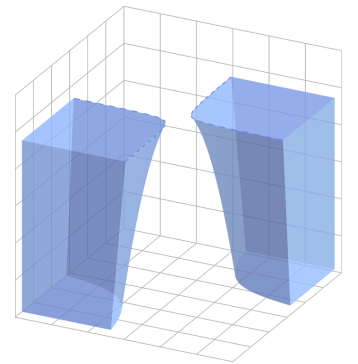
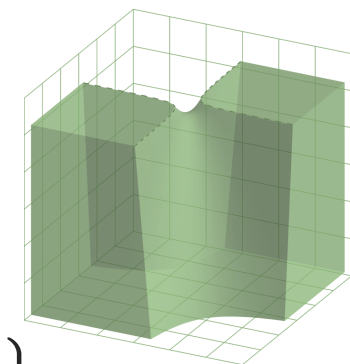
Two connected components

$$\text{GL}_n^+ = \{\Pi \in \mathbb{R}^{n \times n} \mid \det \Pi > 0\},$$

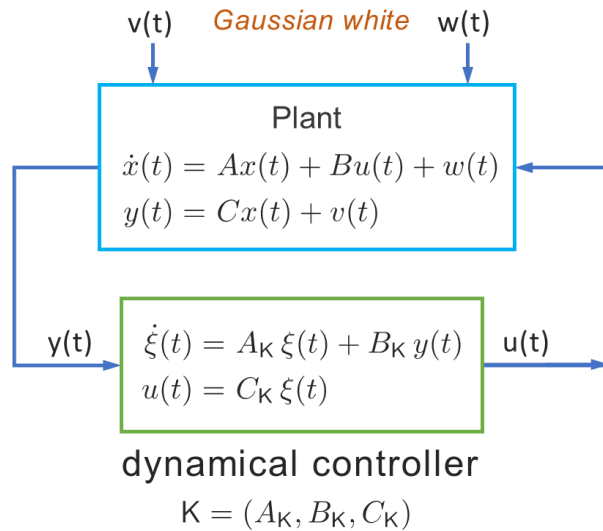
$$\text{GL}_n^- = \{\Pi \in \mathbb{R}^{n \times n} \mid \det \Pi < 0\}.$$

$$\mathcal{F} = \left\{ (X, Y, M, H, F) \mid X, Y \in \mathbb{S}^n, M \in \mathbb{R}^{n \times n}, H \in \mathbb{R}^{n \times p}, F \in \mathbb{R}^{m \times n}, \right.$$

$$\left. \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0, \begin{bmatrix} AX+BF & A \\ M & YA+HC \end{bmatrix} + \begin{bmatrix} AX+BF & A \\ M & YA+HC \end{bmatrix}^\top \prec 0 \right\}$$



# Policy Optimization formulation



## Policy optimization for LQG

$$\min_K J(K)$$

$$\text{s.t. } K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$$

Non-convex  
Landscape  
Analysis

- **Q1: Connectivity of the feasible region  $\mathcal{C}_{\text{full}}$** 
  - Is it connected? **No**
  - If not, how many connected components can it have? **Two**
- **Q2: Structure of stationary points of  $J(K)$** 
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?
- **Q3: How to escape high-order saddle points via PGD?**

# Outline

- LQG problem Setup
- Connectivity of the Set of Stabilizing Controllers
- Structure of Stationary Points of the LQG cost**
- Escaping saddle points via PGD

# Structure of Stationary Points

## □ Simple observations

- 1)  $J(K)$  is a real analytic function over its domain (smooth, infinitely differentiable)
- 2)  $J(K)$  has **non-unique** and **non-isolated** global optima

$$\begin{aligned}\dot{\xi}(t) &= A_K \xi(t) + B_K y(t) \\ u(t) &= C_K \xi(t)\end{aligned}$$

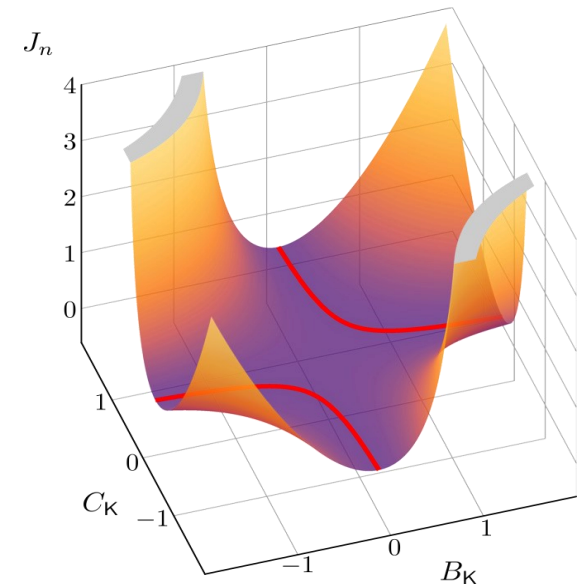
## Similarity transformation

$$(A_K, B_K, C_K) \mapsto (T A_K T^{-1}, T B_K, C_K T^{-1})$$

- $J(K)$  is invariant under similarity transformations.
- It has many stationary points, unlike the LQR with a unique stationary point

## Policy optimization for LQG

$$\begin{aligned}\min_K \quad & J(K) \\ \text{s.t.} \quad & K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}\end{aligned}$$





# Structure of Stationary Points

## □ Gradient computation

**Lemma 2:** For every  $K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$ , we have

$$\frac{\partial J(K)}{\partial A_K} = 2 (Y_{12}^T X_{12} + Y_{22} X_{22}),$$

$$\frac{\partial J(K)}{\partial B_K} = 2 (Y_{22} B_K V + Y_{22} X_{12}^T C^T + Y_{12}^T X_{11} C^T),$$

$$\frac{\partial J(K)}{\partial C_K} = 2 (R C_K X_{22} + B^T Y_{11} X_{12} + B^T Y_{12} X_{22}),$$

where  $X_K = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$ ,  $Y_K = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$

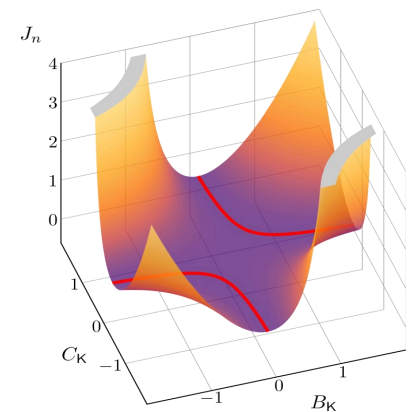
are the unique positive semidefinite solutions to two Lyapunov equations.

**How does the set of Stationary Points look like?**

$$\left\{ K \in \mathcal{C}_{\text{full}} \left| \begin{array}{l} \frac{\partial J(K)}{\partial A_K} = 0, \\ \frac{\partial J(K)}{\partial B_K} = 0, \\ \frac{\partial J(K)}{\partial C_K} = 0, \end{array} \right. \right\}$$

□ Non-unique, non-isolated

□ Local minimum, local maximum, saddle points, or globally minimum?



# Structure of Stationary Points

## □ Main Result: existences of strict saddle points

**Theorem 4:** Consider any open-loop stable plant. The zero controller with any stable  $A_K$

$$K = (A_K, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (**strict saddle point**) or equal to zero (**high-order saddle or else**).

**Example:**

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$

$$Q = 1, R = 1, V = 1, W = 1$$

$$y(t) = x(t) + v(t)$$

$$\text{Stationary point: } K^* = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \text{with } a < 0$$

➤ **Cost function:**  $J\left(\begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix}\right) = \frac{A_K^2 - A_K(1 + B_K^2 C_K^2) - B_K C_K(1 - 3B_K C_K + B_K^2 C_K^2)}{2(-1 + A_K)(A_K + B_K C_K)}$ .

➤ **Hessian:** 
$$\left[ \begin{array}{ccc} \frac{\partial J^2(K)}{\partial A_K^2} & \frac{\partial J^2(K)}{\partial A_K \partial B_K} & \frac{\partial J^2(K)}{\partial A_K \partial C_K} \\ \frac{\partial J^2(K)}{\partial B_K A_K} & \frac{\partial J^2(K)}{\partial B_K^2} & \frac{\partial J^2(K)}{\partial B_K \partial C_K} \\ \frac{\partial J^2(K)}{\partial C_K A_K} & \frac{\partial J^2(K)}{\partial C_K B_K} & \frac{\partial J^2(K)}{\partial C_K^2} \end{array} \right] \Bigg|_{K^* = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}} = \frac{1}{2(1-a)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

**Indefinite with eigenvalues:**

$$0 \text{ and } \pm \frac{1}{2(1-a)}$$

# Structure of Stationary Points

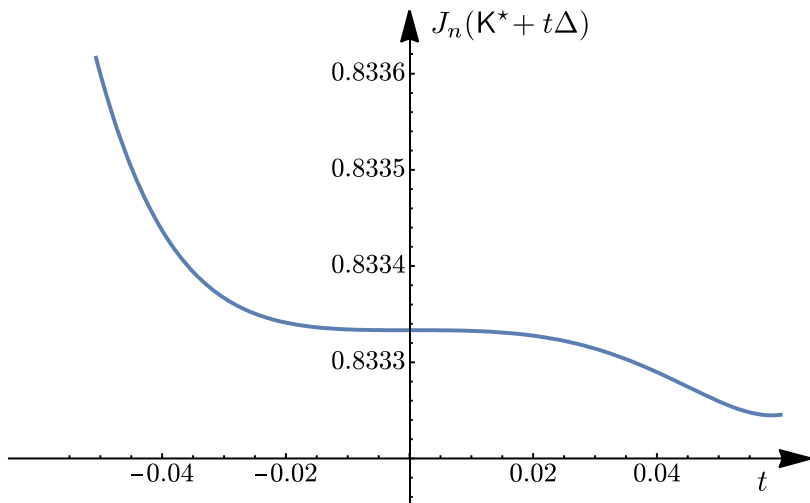
## □ Main Result: existences of strict saddle points

**Theorem 4:** Consider any open-loop stable plant. The zero controller with any stable  $A_K$

$$K = (A_K, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (**strict saddle point**) or equal to zero (**high-order saddle or else**).

## Another example with zero Hessian



## How does the set of Stationary Points look like?

$$\left\{ K \in \mathcal{C}_{\text{full}} \left| \begin{array}{l} \frac{\partial J(K)}{\partial A_K} = 0, \\ \frac{\partial J(K)}{\partial B_K} = 0, \\ \frac{\partial J(K)}{\partial C_K} = 0, \end{array} \right. \right\}$$

□ Non-unique, non-isolated

□ **Strictly suboptimal points; Strict saddle points**

□ All bad stationary points correspond to non-minimal controllers

# Structure of Stationary Points

## □ Main Result

**Theorem 5:**

All stationary points corresponding to controllable and observable controllers are globally optimum.

$$\left\{ K \in \mathcal{C}_{\text{full}} \left| \begin{array}{l} \frac{\partial J(K)}{\partial A_K} = 0, \\ \frac{\partial J(K)}{\partial B_K} = 0, \\ \frac{\partial J(K)}{\partial C_K} = 0, \end{array} \right. \right\}$$

Local Zero Gradient



Structural Information



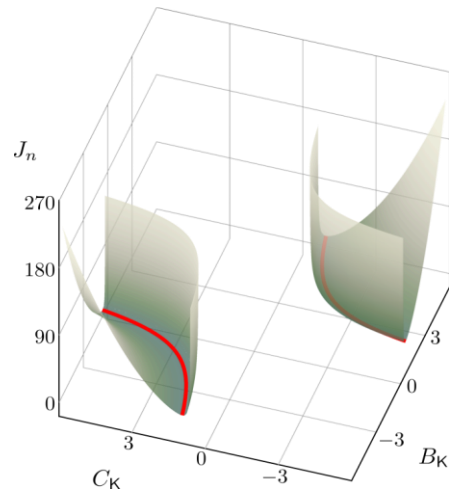
Global Optimality Certificate

Particularly, given a stationary point that is a **minimal** controller

1) It is **globally optimal**, and the set of all global optima forms a manifold with 2 connected components.

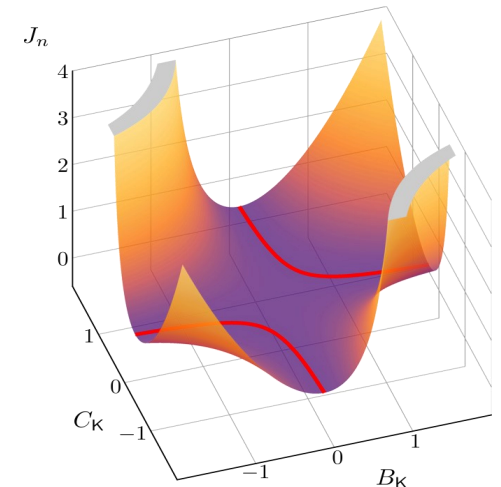
Example: open-loop unstable system

$$\begin{aligned} \dot{x}(t) &= x(t) + u(t) + w(t) \\ y(t) &= x(t) + v(t) \end{aligned}$$



Example: open-loop stable system

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) + w(t) \\ y(t) &= x(t) + v(t) \end{aligned}$$



# Proof idea

## □ Proof: all minimal stationary points are unique up to a similarity transformation

All minimal stationary points  $K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$  to the LQG problem are in the form of

$$A_K = T(A - BK - LC)T^{-1}, \quad B_K = -TL, \quad C_K = KT^{-1},$$

$$K = R^{-1}B^T S, \quad L = PC^T V^{-1},$$

$T$  is an invertible matrix and  $P, S$  are the unique positive definite solutions to the Riccati equations

$$\left\{ K \in \mathcal{C}_{\text{full}} \left| \begin{array}{l} \frac{\partial J(K)}{\partial A_K} = 0, \\ \frac{\partial J(K)}{\partial B_K} = 0, \\ \frac{\partial J(K)}{\partial C_K} = 0, \end{array} \right. \right\} \xrightarrow{\text{Minimal controller}} \begin{array}{l} \frac{\partial J_n(K)}{\partial B_K} = 0 \implies B_K = -TPC^T V^{-1} \\ \frac{\partial J_n(K)}{\partial C_K} = 0 \implies C_K = R^{-1}B^T S T^{-1} \\ \frac{\partial J(K)}{\partial A_K} = 0 \implies A_K = T(A - PC^T V^{-1}C - BR^{-1}B^T S)T^{-1} \end{array}$$

$$\frac{\partial J(K)}{\partial A_K} = 2(Y_{12}^T X_{12} + Y_{22} X_{22}),$$

$$\frac{\partial J(K)}{\partial B_K} = 2(Y_{22} B_K V + Y_{22} X_{12}^T C^T + Y_{12}^T X_{11} C^T),$$

$$\frac{\partial J(K)}{\partial C_K} = 2(R C_K X_{22} + B^T Y_{11} X_{12} + B^T Y_{12} X_{22}),$$

Special case in Theorem 20.6 of Zhou et al., 1996 and  
Section II of Hyland, 1984

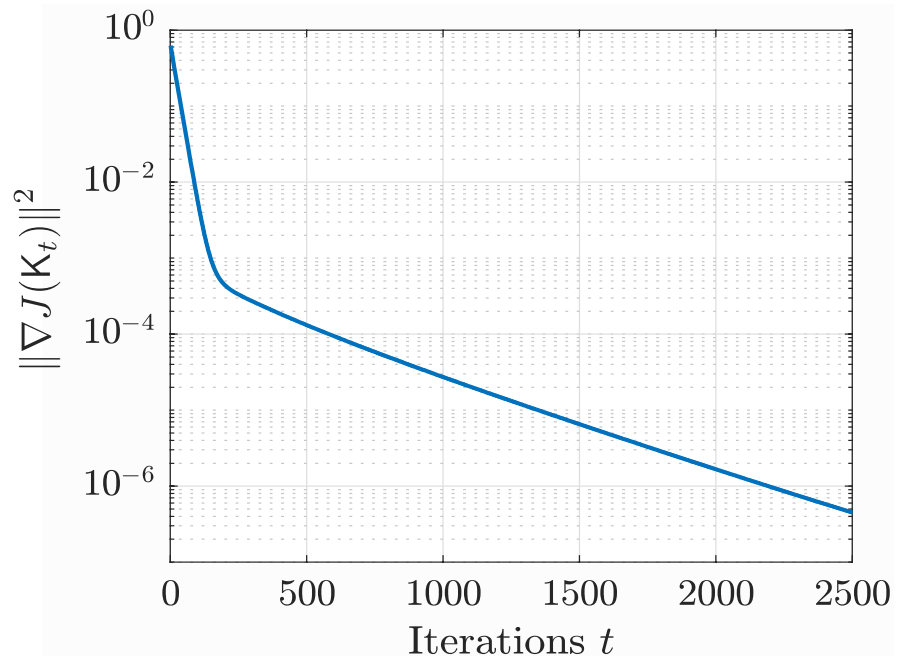
# Structure of Stationary Points

## □ Implication

**Corollary:** Consider gradient descent iterations

$$\mathbf{K}_{t+1} = \mathbf{K}_t - \alpha \nabla J(\mathbf{K}_t)$$

If the iterates converge to a minimal controller, then this minimal controller is a global optima.



### More questions:

- ✓ Escaping saddle points?
- ✓ Convergence conditions?
- ✓ Convergence speed?
- ✓ Alternative model-free parameterization?

A recent and related paper on output estimation is

Umenberger, J., Simchowicz, M., Perdomo, J. C., Zhang, K., & Tedrake, R. (2022). Globally Convergent Policy Search over Dynamic Filters for Output Estimation. *arXiv preprint arXiv:2202.11659*.

# Comparison with LQR

## Policy optimization for LQR

$$\begin{aligned} \min_K & J(K) \\ \text{s.t.} & K \in \mathcal{K} \end{aligned}$$

## Policy optimization for LQG

$$\begin{aligned} \min_K & J(K) \\ \text{s.t.} & K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}} \end{aligned}$$

### Connectivity of feasible region

- ❖ Always connected

- ❖ Disconnected, but at most 2 connected comp.
- ❖ They are almost identical to each other

### Stationary points

- ❖ Unique

- ❖ Non-unique, non-isolated stationary points
- ❖ Spurious stationary points (strict saddle, nonminimal controller)
- ❖ **All mini. stationary points are globally optimal**

### Gradient Descent

- ❖ Gradient dominance
- ❖ Global fast convergence (like strictly convex)

- ❖ No gradient dominance
- ❖ Local convergence/speed (**unknown**)
- ❖ **Many open questions**

### References

Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Frieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others

Zheng\*, Tang\*, Li. 2021, [link](#) (\* equal contribution)

# Outline

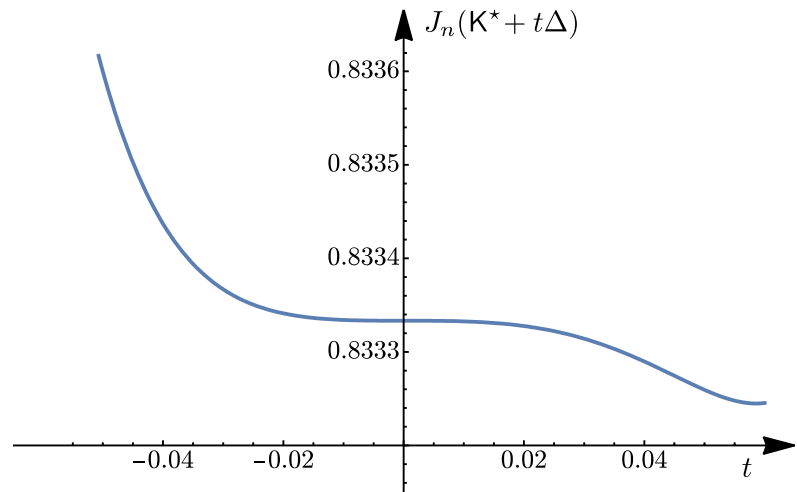
- LQG problem Setup
- Connectivity of the Set of Stabilizing Controllers
- Structure of Stationary Points of the LQG cost
- **Escaping saddle points via PGD**



# Saddle points

## Policy Optimization for LQG

$$\begin{aligned} \min_{\mathbf{K}} \quad & J(\mathbf{K}) \\ \text{s.t.} \quad & \mathbf{K} = (A_{\mathbf{K}}, B_{\mathbf{K}}, C_{\mathbf{K}}) \in \mathcal{C}_{\text{full}} \end{aligned}$$



## Local geometry

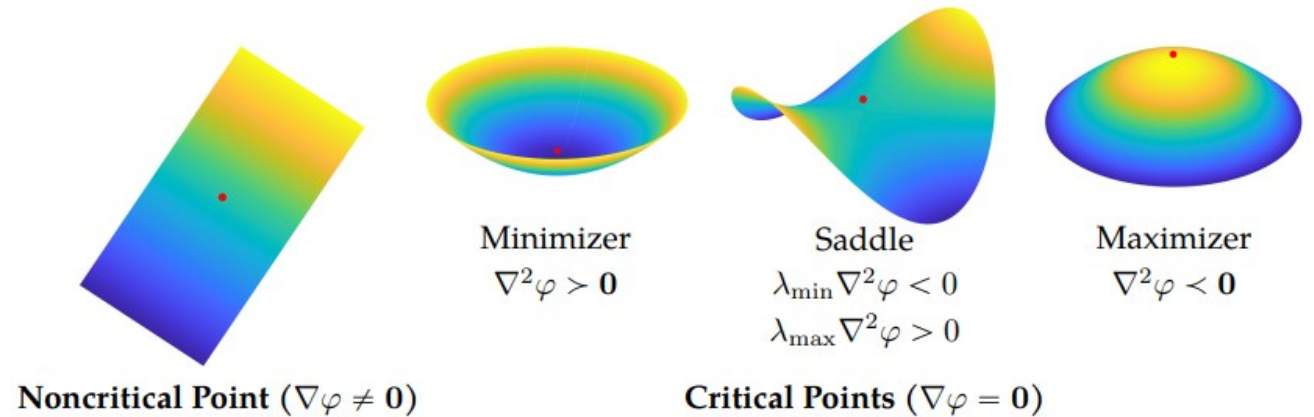


Figure taken from Zhang et al., 2020

- ❑ **Strict saddle points:** the hessian has a strict negative eigenvalue (i.e., escaping direction)
- ❑ **Non-strict (high-order) saddle points:** no such escaping direction, i.e., minimum eigenvalue is zero.
- ❑ **Simple perturbed gradient descent (PGD)** methods can escape strict saddle points efficiently (e.g., Jin et al., 2017)

# Structure of stationary points

□ **Theorem (informal):** all bad stationary points are in the same form

$$\left\{ K \in \mathcal{C}_n \left[ \begin{array}{l} \frac{\partial J(K)}{\partial A_K} = 0, \\ \frac{\partial J(K)}{\partial B_K} = 0, \\ \frac{\partial J(K)}{\partial C_K} = 0, \end{array} \right. \right.$$

a stationary point

$$K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathcal{C}_n$$

If it is minimal, then it is globally optimal

If it is not minimal, find a minimal realization

$$\hat{K} = \begin{bmatrix} 0 & \hat{C}_K \\ \hat{B}_K & \hat{A}_K \end{bmatrix} \in \mathcal{C}_q$$

The following full-order controller with any stable  $\Lambda$  is also a stationary point with the same LQG cost

$$\tilde{K} = \begin{bmatrix} 0 & \hat{C}_K & 0 \\ \hat{B}_K & \hat{A}_K & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \in \mathcal{C}_n$$

$$\dot{\xi}(t) = A_K \xi(t) + B_K y(t),$$

$$u(t) = C_K \xi(t),$$

$$\dot{\hat{\xi}}(t) = \begin{bmatrix} \hat{A}_K & 0 \\ 0 & \Lambda \end{bmatrix} \hat{\xi}(t) + \begin{bmatrix} \hat{B}_K \\ 0 \end{bmatrix} y(t),$$

$$u(t) = [\hat{C}_K \quad 0] \hat{\xi}(t),$$

where we isolate the uncontrollable and unobservable part

# Strict saddle points

$$\left\{ K \in \mathcal{C}_n \left| \begin{array}{l} \frac{\partial J(K)}{\partial A_K} = 0, \\ \frac{\partial J(K)}{\partial B_K} = 0, \\ \frac{\partial J(K)}{\partial C_K} = 0, \end{array} \right. \right.$$

a stationary point

$$K = \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix} \in \mathcal{C}_n$$



The same form

$$\tilde{K} = \begin{bmatrix} 0 & \hat{C}_K & 0 \\ \bar{B}_K & \hat{A}_K & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \in \mathcal{C}_n$$

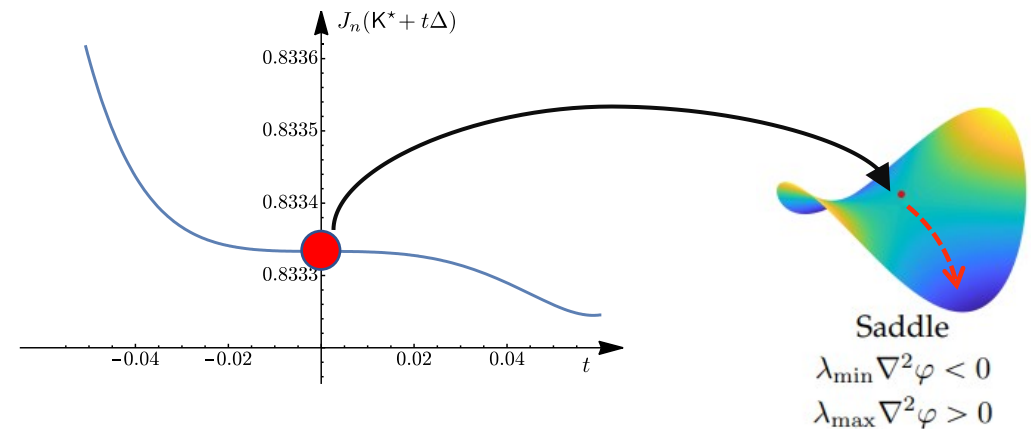
□ **Theorem (informal):** Under a mild condition, choosing the diagonal stable block  $\Lambda$  randomly leads to a strict saddle point with probability 1

## Our idea: a structural perturbation

A high-order saddle



A strict saddle point with the same LQG cost



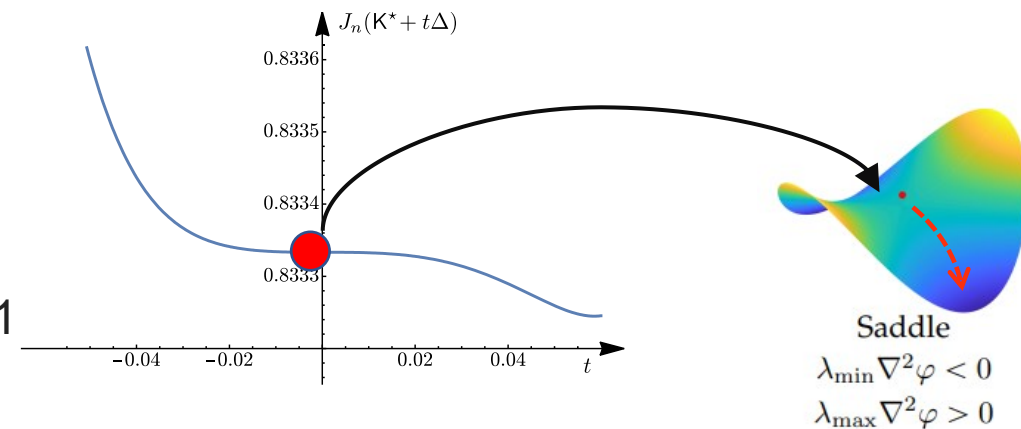
✓ Yang Zheng\*, Yue Sun\*, Maryam Fazel, and Na Li. "Escaping High-order Saddles in Policy Optimization for Linear Quadratic Gaussian (LQG) Control." arXiv preprint arXiv:2204.00912 (2022). \*Equal contribution

# Perturbed Gradient Descent

□ **Theorem (informal):** all bad stationary points are in the same form

$$\tilde{K} = \begin{bmatrix} 0 & \hat{C}_K & 0 \\ \tilde{B}_K & \hat{A}_K & 0 \\ 0 & 0 & \Lambda \end{bmatrix} \in \mathcal{C}_n$$

□ **Theorem (informal):** Choosing the diagonal stable block  $\Lambda$  randomly leads to a strict saddle point with probability almost 1



**Our idea: a structural perturbation + standard PGD**

A non-optimal stationary point



A strict saddle point with the same LQG cost

Perturbation on  $\Lambda$



Standard PGD algorithm (Jin et al., 2017)

Perturbation on gradients

# Numerical simulations

## Three policy gradient algorithms

1. Vanilla gradient descent  $K_{i+1} = K_i - \alpha_i \nabla J(K_i)$
2. Standard PGD algorithm (adding a small random perturbation on iterates; Jin et al., 2017;)
3. **Structural perturbation + standard PGD**

**Example:** System dynamics

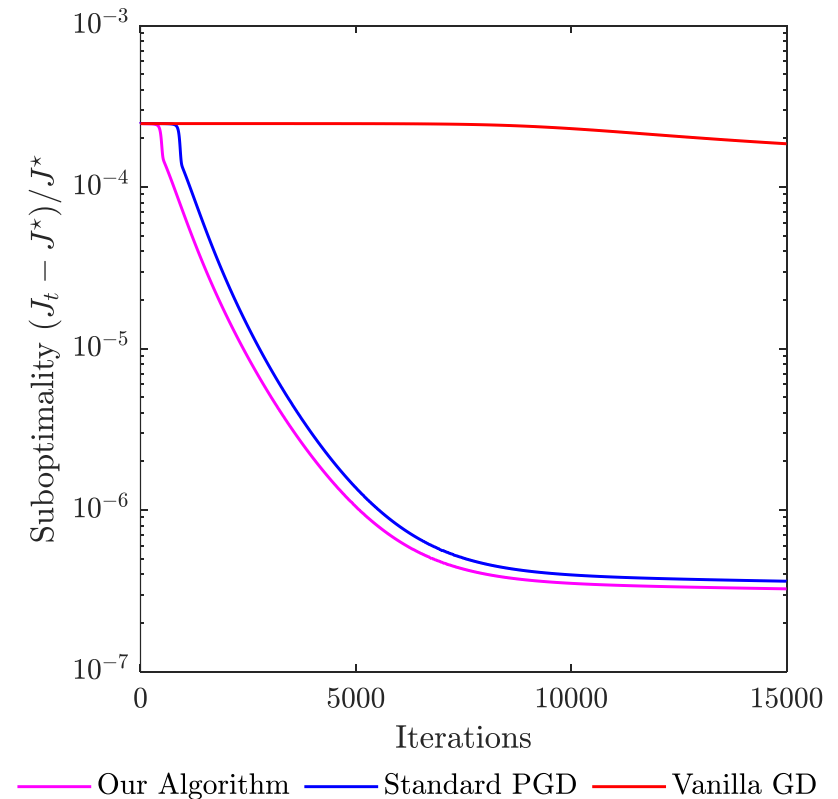
$$A = \begin{bmatrix} -0.5 & 0 \\ 0.5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -\frac{1}{6} & \frac{11}{12} \end{bmatrix},$$

Performance weights

$$W = Q = I_2, \quad V = R = 1$$

A point that is close to a high-order saddle with zero hessian

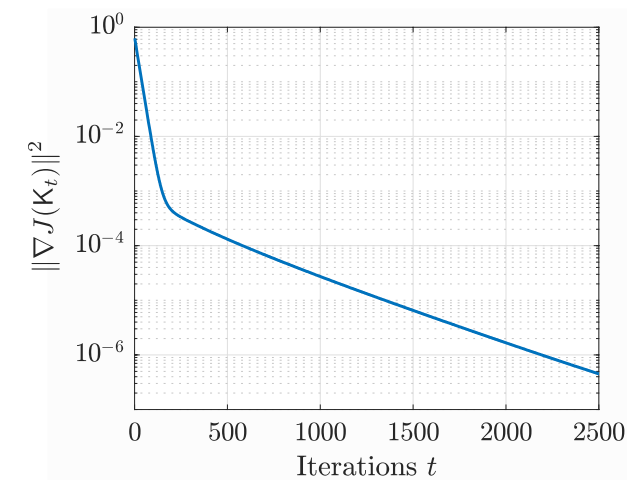
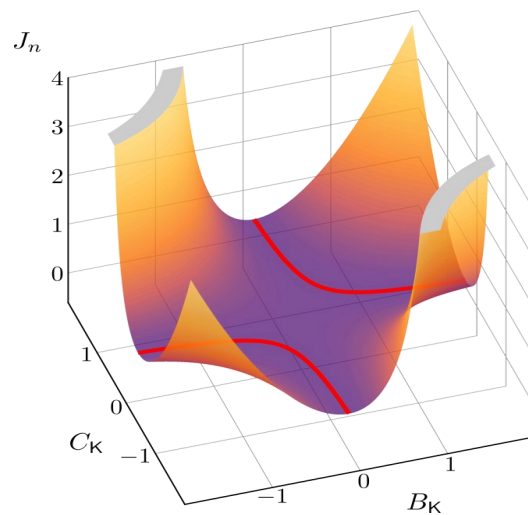
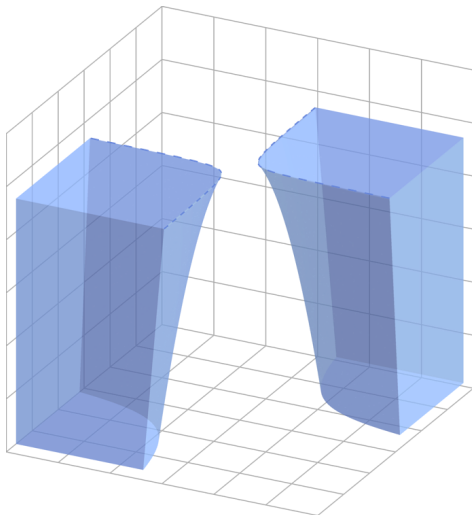
$$A_{K,0} = -0.5I_2, \quad B_{K,0} = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \quad C_{K,0} = [0, -0.01]$$



# Conclusions

# Nonconvex optimization for LQG control

- ❑ Much richer and more complicated than LQR
- ❑ Disconnected, but at most 2 connected components
- ❑ Non-unique, non-isolated stationary points, strict saddle points
- ❑ Minimal (controllable and observable) stationary points are globally optimal
- ❑ A new perturbed gradient descent algorithm



# Ongoing and Future work

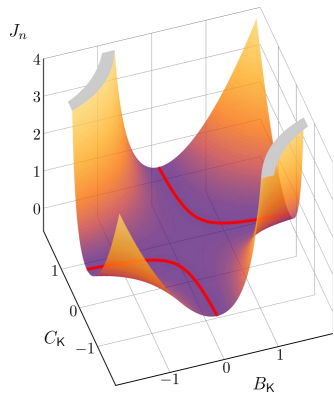
- ❑ How to certify the optimality of a non-minimal stationary point
- ❑ Convergence proof of perturbed policy gradient (PGD)
- ❑ More quantitative analysis of PGD algorithms for LQG
- ❑ Alternative model-free parametrization of dynamical controllers (e.g., Makdah & Pasqualetti, 2023; Zhao, Fu & You, 2022.)
  - ✓ Better optimization landscape structures, smaller dimension
- ❑ Nonconvex Landscape of  $H_{\infty}$  dynamical output feedback control (Tang & Zheng, 2023 <https://arxiv.org/abs/2304.00753>; )



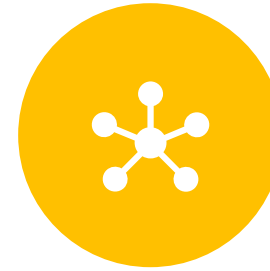
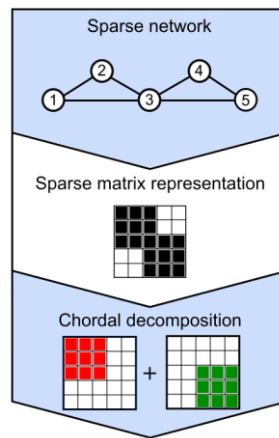
# SOC lab at UC San Diego



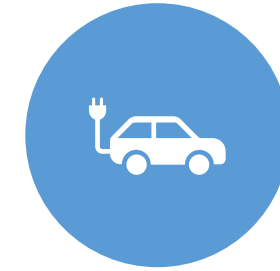
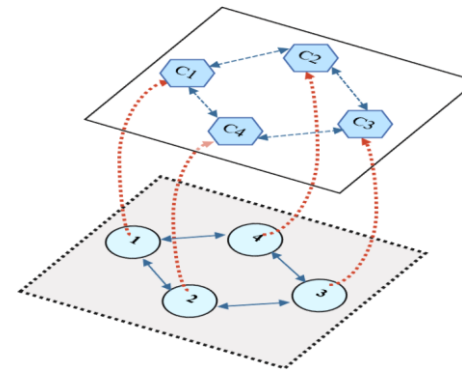
**Data-driven and learning-based control**



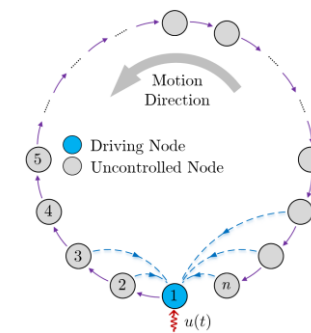
**Sparse conic optimization**



**Scalable distributed control**



**Connected and autonomous vehicles (CAVs)**



Check out our webpage: <https://zhengy09.github.io/soclab.html>

# Nonconvex Optimization for Linear Quadratic Gaussian (LQG) Control

Thank you for your attention!

Q & A

1. Y. Tang\*, Y. Zheng\*, and N. Li, "Analysis of the optimization landscape of Linear Quadratic Gaussian (LQG) control," Mathematical Programming, 2023. Available: <https://arxiv.org/abs/2102.04393> \*Equal contribution
2. Y. Zheng\*, Y. Sun\*, M. Fazel, and N. Li. "Escaping High-order Saddles in Policy Optimization for Linear Quadratic Gaussian (LQG) Control." arXiv preprint, 2022 <https://arxiv.org/abs/2204.00912>. \*Equal contribution