# Benign Nonconvex Landscapes in Optimal and Robust Control

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## Overview and Nonconvex Landscapes

- **Policy optimization has achieved great empirical success in various applications.**
- **A strong theoretical foundation is lacking for continuous control tasks.**
- **LQR, LQG, H infinity control all have intriguing nonconvex landscapes.**

| LQG optimal control | H infinity robust control |
| --- | --- |
| $\min_{u} \quad J_{\texttt{LQG}} := \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T (x^TQx + u^TRu)\right]$ <br> s. t. $\dot{x} = Ax + Bu + w$ <br> $y = Cx + v$ <br> $w, v$ white Gaussian noises | $\min_{u} \quad J_\infty := \max_{\Vert d\Vert\leq 1} \int_0^\infty (x^TQx + u^TRu)dt$ <br> s. t. $\dot{x} = Ax + Bu + w, \; x(0) = 0$ <br> $y = Cx + v$ <br> $d = [w^T \quad v^T]^T$ |

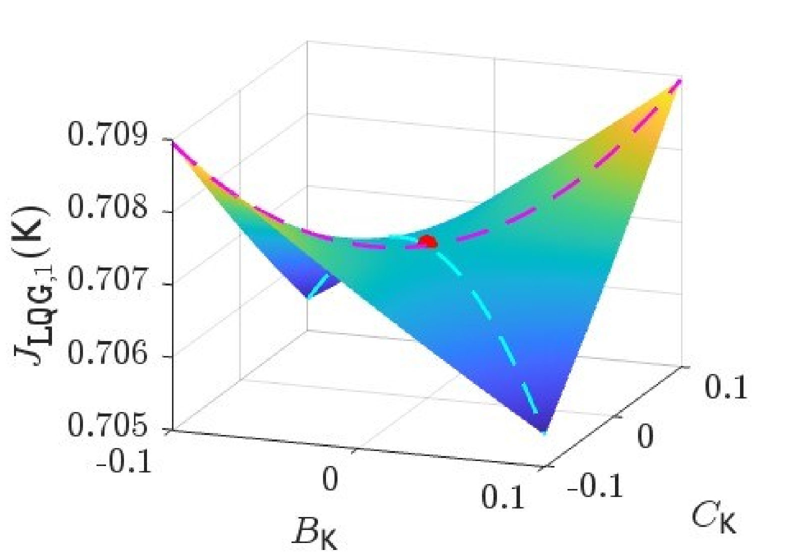
$$\min_{K\in\mathcal{K}} J(K)$$

**Dynamic policies**

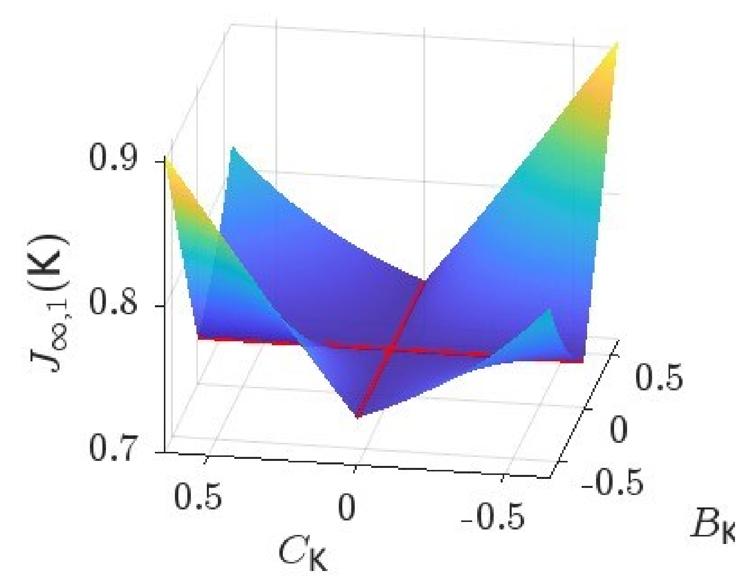
$$\mathsf{K} = \begin{bmatrix} D_{\mathsf{K}} & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix}$$

$$\dot{\hat{x}} = A_{\mathsf{K}}\hat{x} + B_{\mathsf{K}}y$$
$$u = C_{\mathsf{K}}\hat{x} + D_{\mathsf{K}}y$$

**LQG: saddle point**




**H infinity: non-smooth**




## Structure of Stationary Points

**Theorem 1** **Any non-degenerate Clarke stationary point is globally optimal.**

## Key Ideas

- **Non-strict Linear Matrix Inequality (LMI)**

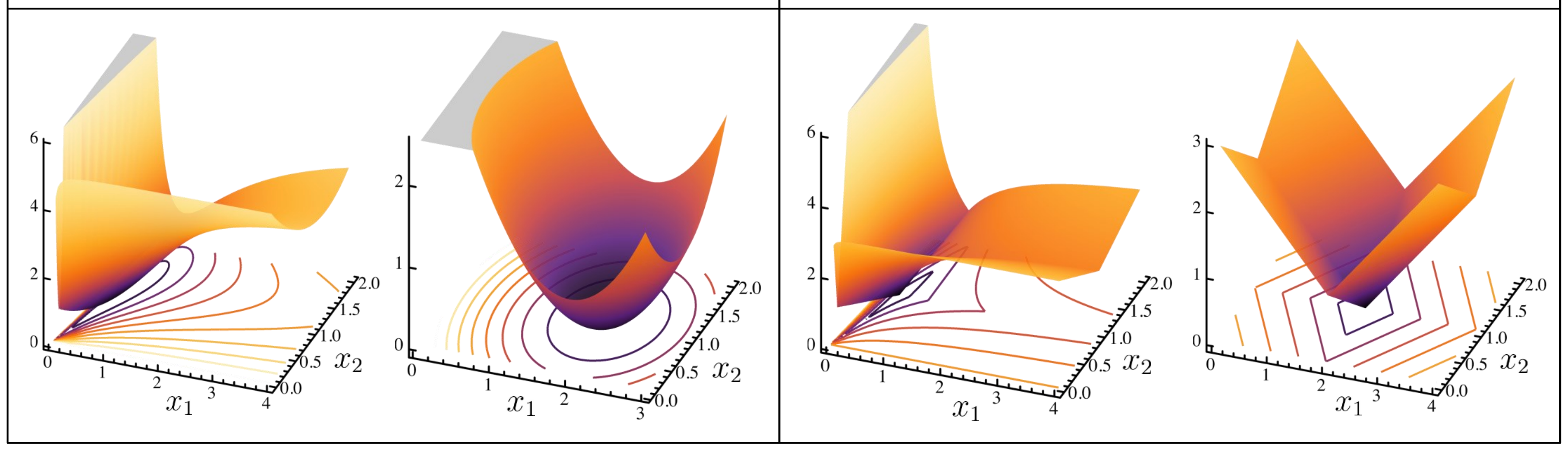
$$J_{\texttt{LQG},n}(\mathsf{K}) \leq \gamma \text{ if } \exists P,\Gamma \text{ s.t.} \begin{bmatrix} A_{\rm cl}(\mathsf{K})^{\mathsf{T}}P + PA_{\rm cl}(\mathsf{K}) & PB_{\rm cl}(\mathsf{K}) \\ B_{\rm cl}(\mathsf{K})^{\mathsf{T}}P & -\gamma I \end{bmatrix} \preceq 0,$$
$$\begin{bmatrix} P & C_{\rm cl}(\mathsf{K})^{\mathsf{T}} \\ C_{\rm cl}(\mathsf{K}) & \Gamma \end{bmatrix} \succeq 0, \; P \succ 0, \; \mathrm{trace}(\Gamma) \leq \gamma.$$

- **Convex Reformulation with change of variables**

**when** $\gamma = J_{\texttt{LQG},n}(\mathsf{K})$ **and** $P_{12} \succ 0$, where $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathsf{T}} & P_{22} \end{bmatrix}$

## Unified Framework: Extended Convex Lifting (ECL)

| | |
| --- | --- |
| $f_1(x_1,x_2) = \left(\frac{x_2}{x_1} - 2\right)^2 + (x_2-1)^2,$ <br> $h_1(y_1,y_2) = (y_1-2)^2 + (y_2-1)^2$ | $f_2(x_1,x_2) = \left\vert\frac{x_2}{x_1} - 2\right\vert + \vert x_2 - 1\vert$ <br> $h_2(y_1,y_2) = \vert y_1 - 2\vert + \vert y_2 - 1\vert$ |




The tuple $(\mathcal{L}_{\rm lft}, \mathcal{F}_{\rm cvx}, \mathcal{G}_{\rm aux}, \Phi)$ is an ECL of $f$ if

1) $\mathcal{L}_{\rm lft} \subseteq \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d_\xi}$ is a lifted set with an extra variable $\xi \in \mathbb{R}^{d_\xi}$, such that the canonical projection of $\mathcal{L}_{\rm lft}$ onto the first $d+1$ coordinates, given by $\pi_{x,\gamma}(\mathcal{L}_{\rm lft}) = \{(x,\gamma) : \exists \xi \in \mathbb{R}^{d_\xi} \text{ s.t. } (x,\gamma,\xi) \in \mathcal{L}_{\rm lft}\}$, satisfies
$$\mathrm{epi}_>(f) \subseteq \pi_{x,\gamma}(\mathcal{L}_{\rm lft}) \subseteq \mathrm{cl}\,\mathrm{epi}_\geq(f). \tag{6a}$$
2) $\mathcal{F}_{\rm cvx} \subseteq \mathbb{R} \times \mathbb{R}^{d_1}$ is a convex set, $\mathcal{G}_{\rm aux} \subseteq \mathbb{R}^{d_2}$ is an auxiliary set, and $\Phi$ is a $C^2$-diffeomorphism from $\mathcal{L}_{\rm lft}$ to $\mathcal{F}_{\rm cvx} \times \mathcal{G}_{\rm aux}$.[3]
3) For any $(x,\gamma,\xi) \in \mathcal{L}_{\rm lft}$, we have
$$\Phi(x,\gamma,\xi) = (\gamma,\zeta_1,\zeta_2) \quad \text{and} \quad (\gamma,\zeta_1) \in \mathcal{F}_{\rm cvx} \tag{6b}$$
for some $\zeta_1 \in \mathbb{R}^{d_1}$ and $\zeta_2 \in \mathcal{G}_{\rm aux}$ (i.e., the map $\Phi$ directly outputs $\gamma$ in the first component).

**Non-degenerate points** $x \in \mathcal{D}_{\rm nd} : (x, f(x)) \in \pi_{x,\gamma}(\mathcal{L}_{\rm lft})$

**Theorem 2** **Convex reformulation** $\inf_{x\in\mathcal{D}} f(x) = \inf_{(\gamma,\zeta_1)\in\mathcal{F}_{\rm cvx}} \gamma.$

**Theorem 3** $x^* \in \mathcal{D}_{\rm nd}$ and $0 \in \partial f(x^*) \Rightarrow f(x^*) = \inf_{x\in\mathcal{D}} f(x)$

**No spurious stationary points**

## References & Acknowledgement


- ❑ Yang Zheng, Chih-fan Pai, and Yujie Tang. "Benign Nonconvex Landscapes in Optimal and Robust Control, Part I: Global Optimality." *arXiv:2312.15332*, 2023.
- ❑ Zheng, Yang, Chih-Fan Pai, and Yujie Tang. "Benign Nonconvex Landscapes in Optimal and Robust Control, Part II: Extended Convex Lifting." *arXiv:2406.04001* (2024).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