# Large-scale Autonomy: Scalable Sparse Optimization and Distributed Control


Yang Zheng [1,2]

[1]School of Engineering and Applied Sciences, Harvard University [2]Harvard Center for Green Buildings and Cities, Harvard University


## 1. Graphs & Decomposition

Exploiting underlying structures (*e.g.*, low rank, symmetry, and sparsity) is one key direction to improve the scalability of solving convex optimization.

- An undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is chordal if every cycle of length greater than 3 has a chord.
- Sparse positive semidefinite matrices $\mathbb{S}^n_+(\mathcal{E},0)$.
- For chordal graphs, we have

$$Z \in \mathbb{S}^n_+(\mathcal{E},0) \Leftrightarrow Z = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} Z_k E_{\mathcal{C}_k}, Z_k \in \mathbb{S}^{|\mathcal{C}_k|}_+.$$

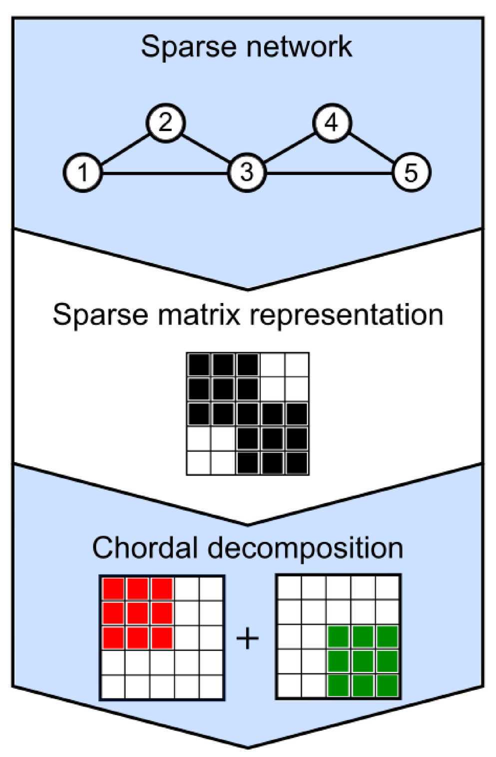



Figure: Chordal decomposition of a sparse network.

## 2 Sparse Semidefinite Programs

Consider a sparse SDP in the dual form

$$\max_{y,\,Z} \quad \langle b, y\rangle$$
$$\text{subject to} \quad Z + \sum_{i=1}^{m} A_i\, y_i = C,$$
$$Z \in \mathbb{S}^n_+,$$

where $A_i, C \in \mathbb{S}^n(\mathcal{E},0)$. This can be equivalently reformulated as

$$\max_{y, Z_k, V_k} \quad \langle b, y\rangle$$
$$\text{subject to} \quad \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} V_k E_{\mathcal{C}_k} + \sum_{i=1}^{m} A_i\, y_i = C,$$
$$Z_k - V_k = 0,\ k = 1,\ldots,t,$$
$$Z_k \in \mathbb{S}^{|\mathcal{C}_k|}_+, \quad k = 1,\ldots,t,$$

which is referred to as *range-space* decomposition.

- Primal SDPs $\rightarrow$ *domain-space* decomposition.
- Suitable for operator-splitting algorithms [1].
- Cone Decomposition Conic solver (CDCS) `https://github.com/OxfordControl/CDCS`

## The Big Picture

- Many urban systems are currently of a size and complexity that render traditional methods both intractable in theory and unscalable in computation.
- My work focuses on developing **computationally scalable** and **theoretically tractable** methods for control and optimization of large-scale autonomous systems.

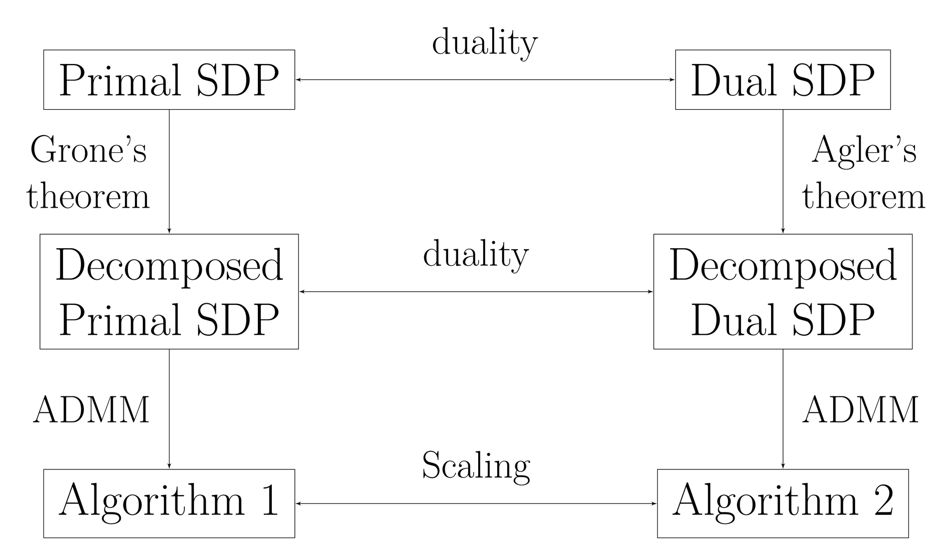



Figure: Duality is inherited in the decomposition and algorithmic development [1]

Table: Average CPU time per iteration (in seconds) for benchmark SDPs.

| | rs35 | rs200 | rs228 | rs365 | rs1555 | rs1907 |
|---|---|---|---|---|---|---|
| SCS (direct) | 1.19 | 4.85 | 1.17 | 17.25 | 69.59 | 25.24 |
| CDCS-primal | 0.94 | 0.26 | 0.22 | 0.72 | 0.83 | 0.83 |
| CDCS-dual | 1.06 | 0.26 | 0.23 | 0.77 | 0.79 | 0.92 |
| CDCS-hsde | 1.01 | 0.22 | 0.21 | 0.74 | 0.68 | 0.89 |

## 3 Sparse SOS Optimization

**Question: how to verify** $p(x) \geq 0, \forall x \in \mathbb{R}^n$**?**

- **Sum-of-squares (SOS) polynomials**: $p(x)$ can be represented as a sum of finite squared polynomials $f_i(x), i = 1,\ldots,m$

$$p(x) = \sum_{i=1}^{m} f_i^2(x),$$

- **SDP characterization** (Parrilo, Lasserre *etc.*): $p(x)$ is SOS if and only if there exists $Q \succeq 0$,

$$p(x) = v_d(x)^T Q v_d(x).$$

where $v_d(x) = [1, x_1, x_2, \ldots, x_n, x_1^2, \ldots, x_n^d]^{\mathsf{T}}$ is the standard monomial basis.

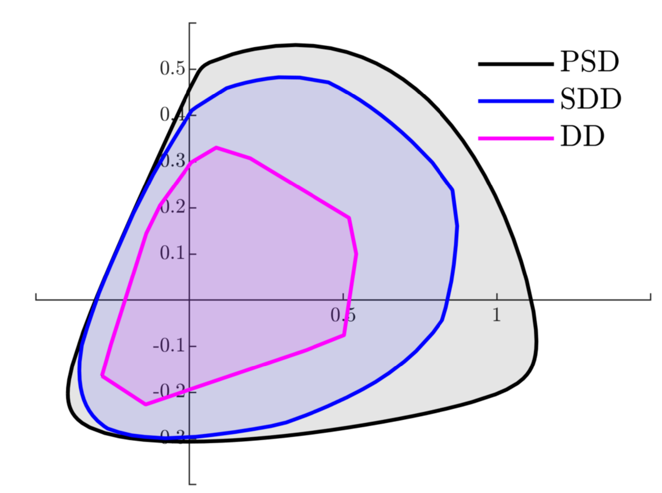
- **Scalability issue:** The size of the resulting SDP is

$$\binom{n+d}{d} \times \binom{n+d}{d},$$

*e.g.* $n = 10, d = 4 \rightarrow 1001$.

A new concept by Ahmadi and Majumdar, 2019

- **SOS:** $p(x) = v_d^{\mathsf{T}} Q v_d$ : $Q$ is PSD $\rightarrow$ SDP
- **SDSOS:** $p(x) = v_d^{\mathsf{T}} Q v_d$ : $Q$ is sdd $\rightarrow$ SOCP
- **DSOS:** $p(x) = v_d^{\mathsf{T}} Q v_d$ : $Q$ is dd $\rightarrow$ LP

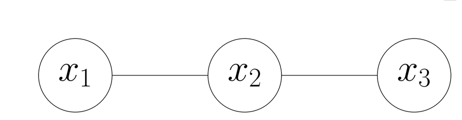



We have proposed two main strategies to bridge the gap

1. Exploit correlative sparsity in $p(x)$ [2];
2. Introduce a new notion of block SDD matrices [3].

Consider $p(x) = x_1^2 + x_1x_2 + x_2x_3^3$ and we define

$$\text{csp}(x_1^2 + x_1x_2 + x_2x_3^3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$




Translate the sparsity into matrix $Q$

- Define a subset $SSOS_{n,2d}(\mathcal{E}) \subset SOS_{n,2d}(\mathcal{E})$ by imposing $Q_{\beta,\gamma} = 0$ if $x^{\beta+\gamma}$ violates the csp $\mathcal{E}$.

Then, we can prove

$$p(x) \in SSOS_{n,2d}(\mathcal{E}) \Longleftrightarrow p(x) = \sum_{k=1}^{t} p_k(E_{\mathcal{C}_k} x).$$

A hierarchy of inner approximations:

$$DSOS(\mathcal{E}) \subset SDSOS(\mathcal{E}) \subset SSOS(\mathcal{E}) \subset SOS(\mathcal{E})$$

and

$DSOS(\mathcal{E}) \longrightarrow$ LP (PSD cones: $1 \times 1$)
$SDSOS(\mathcal{E}) \longrightarrow$ SOCP (PSD cones: $2 \times 2$)
$SSOS(\mathcal{E}) \longrightarrow$ SDP with smaller cones of $k \times k$
$SOS(\mathcal{E}) \longrightarrow$ SDP with a PSD cone of $N \times N$

## 4. Tractable Distributed Control

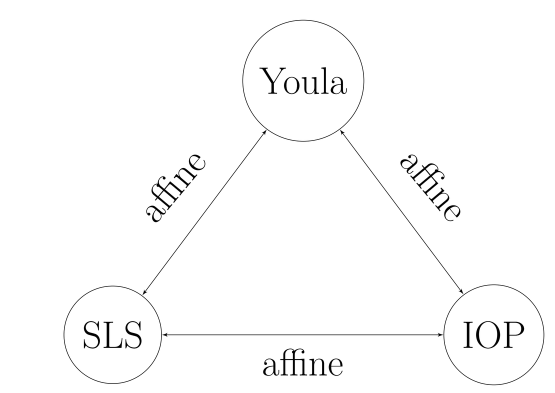
Distributed optimal controller synthesis

$$\min_{\mathbf{K}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{P}_{21}\|$$
$$\text{subject to} \quad \mathbf{K} \text{ internally stabilizes } \mathbf{G},$$
$$\mathbf{K} \in \mathcal{S}.$$

Our contributions

1. A new input-output (IOP) framework and explicit equivalence with Youla and SLS [4].

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{G}\mathbf{K})^{-1} & (I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} \\ \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} & (I - \mathbf{K}\mathbf{G})^{-1} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}.$$




2. A new notion of *Sparsity Invariance* (SI) beyond the well-known *Quadratic Invariance* (QI) [5].

### Sparsity Invariance

- Change of variables: $\mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{S}$
- Design two separate subspaces $\mathcal{T}$ and $\mathcal{R}$

$$\forall\, \mathbf{U} \in \mathcal{T}, \mathbf{Y} \in \mathcal{R} \Rightarrow \mathbf{K} = \mathbf{U}\mathbf{Y}^{-1} \in \mathcal{S}$$

## References


[1] Zheng, Y., Fantuzzi, G., etc. (2019). Chordal decomposition in operator-splitting methods for sparse semidefinite programs. Mathematical Programming, 1-44.

[2] Zheng, Y., Fantuzzi, G., Papachristodoulou, A. (2019). Sparse sum-of-squares (SOS) optimization: A bridge between DSOS/SDSOS and SOS optimization for sparse polynomials. In 2019 American Control Conference (ACC) (pp. 5513-5518). IEEE..

[3] Zheng, Y. Sootla, A., & Papachristodoulou, A. (2019). Block factor-width-two matrices and their applications in semidefinite and sum-of-squares optimization.

[4] Zheng, Y., Furieri, L., Papachristodoulou, A., Li, N., & Kamgarpour, M. (2019). On the Equivalence of Youla, System-level and Input-output Parameterizations. arXiv preprint arXiv:1907.06256. under review at IEE Transactions on Automatic Control.

[5] Furieri, L., Zheng, Y., Papachristodoulou, A., & Kamgarpour, M. (2019). Sparsity Invariance for Convex Design of Distributed Controllers. preprint arXiv:1906.06777.


## Acknowledgements


Y. Zheng was supported by the Clarendon Scholarship and Janson Hu Scholarship at the University of Oxford. He would like to thank the co-authors: A. Papachristodoulou, P. Goulart, A. Wynn, A. Sootla, G. Fantuzzi, M. Kamgarpour, L. Furieri, J. Miller, M. Sznaier, J. Saunderson, G. Hall, A. A. Ahmadi, and N. Li.


Harvard John A. Paulson School of Engineering and Applied Sciences

UNIVERSITY OF OXFORD