

Extended Convex Lifting for Policy Optimization in Control

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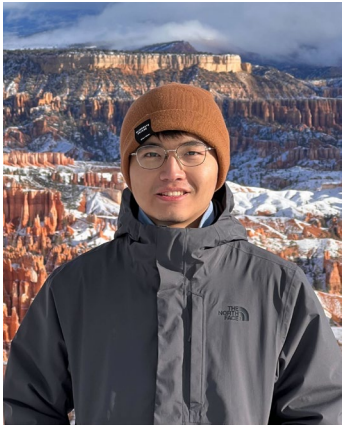
Optimization for ML and AI Seminar

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UC San Diego
JACOBS SCHOOL OF ENGINEERING
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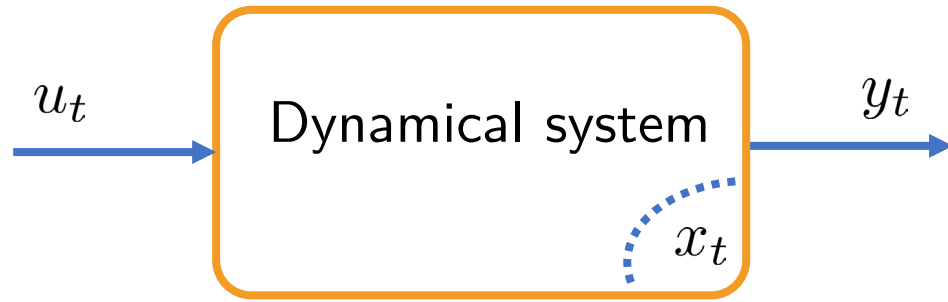
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Introduction

Control theory is the study of dynamical systems with inputs



$$x_{t+1} = f(x_t, u_t)$$

$x_t \in \mathbb{R}^n$ is called the system state

$u_t \in \mathbb{R}^m$ is called the control input

$y_t \in \mathbb{R}^p$ is called the system output

Optimal control

$$\min_{u_t} \sum_{t=0}^T c_t(x_t, u_t)$$

subject to $x_{t+1} = f(x_t, u_t)$

Control as **Optimization** over time
constrained by dynamics

Introduction: a simple control example

Done flying: position control



State $x_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$

Newton's Laws

$$p_{t+1} = p_t + v_t \Delta_t$$

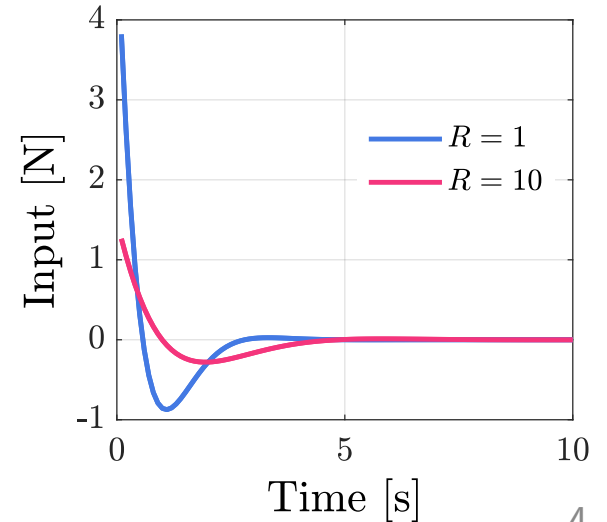
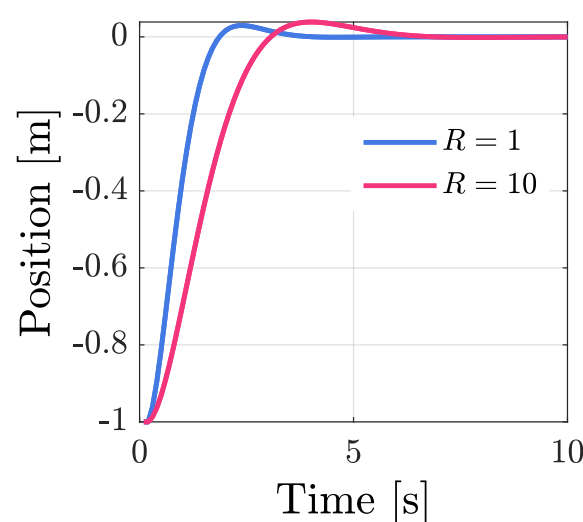
$$v_{t+1} = v_t + a_t \Delta_t$$

$$ma_t = u_t$$

Optimal control

$$\min_{u_t} \sum_{t=0}^T \left(\|x_t\|^2 + R \cdot u_t^2 \right)$$

$$\text{subject to } x_{t+1} = \begin{bmatrix} 1 & \Delta_t \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ \frac{\Delta_t}{m} \end{bmatrix} u_t$$



Many control applications



Temperature/HVAC control



Robotics

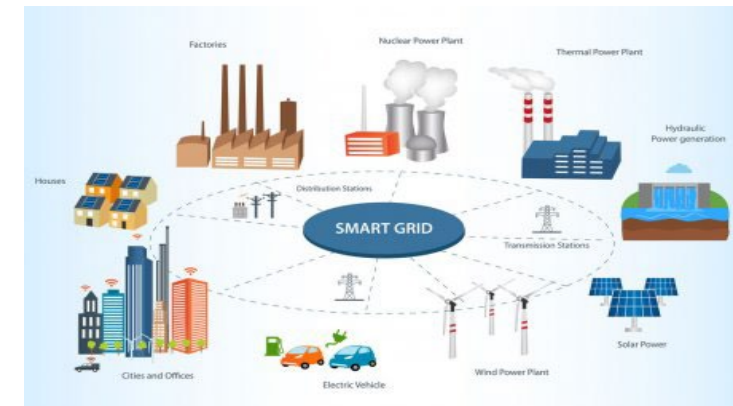


Drone formations



Transportation

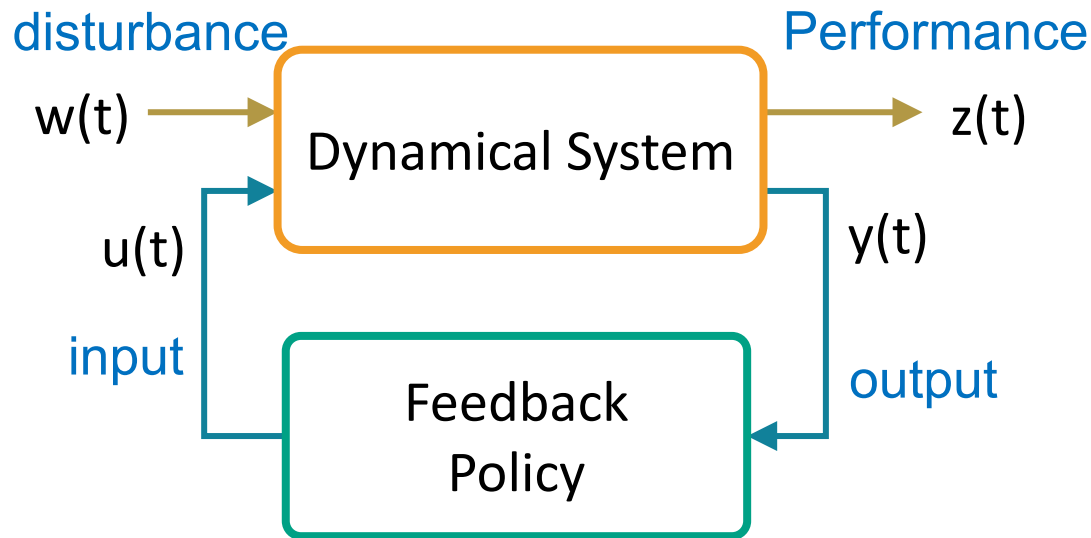
Control theory offers the principled use of **feedback loops** and **algorithms** to drive a dynamic system to its goal.



Smart grid

Today's topic: Policy optimization

Feedback control



- Traditional methods are often **model-based**;
- Well-established theory on **LQR, LQG, Robust control** etc.
- **Tools**: Riccati equations, linear matrix inequalities, Dynamic programming, etc.

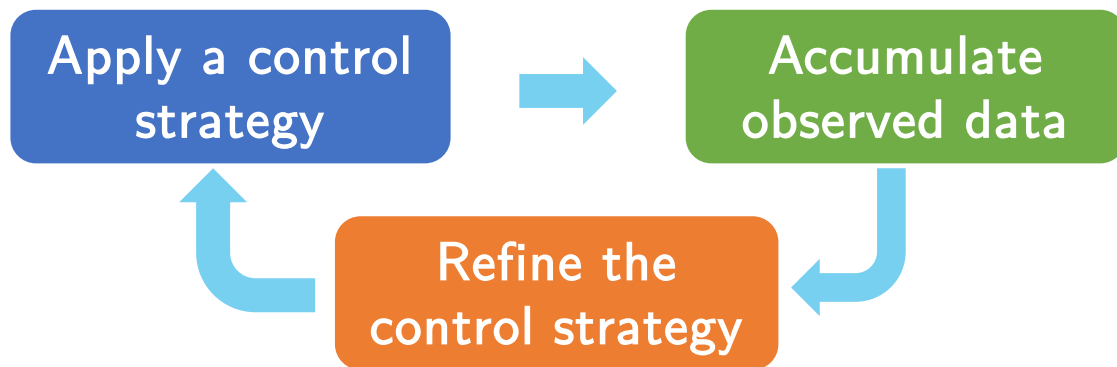
Policy Optimization


Policy parametrization

$$\begin{aligned} \min_{\mathbf{K}} \quad & J(\mathbf{K}) \\ \text{s.t.} \quad & \mathbf{K} \in \mathcal{C} \end{aligned}$$

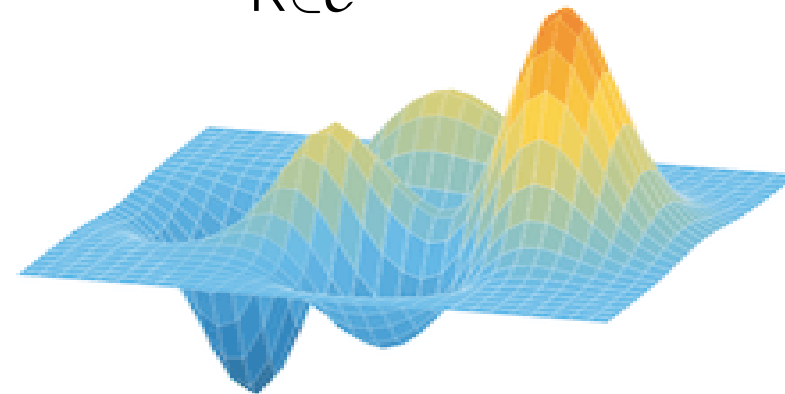
- Use direct policy updates (**model-free**),
$$K_{t+1} = K_t - \alpha_t \nabla J(K_t)$$
- Become very popular in both academia and practice, from **game playing**, **robotics** to **drone flying**, etc.

Today's topic: Policy optimization



$$K_{t+1} = K_t - \alpha_t \nabla J(K_t)$$

$$\min_{K \in \mathcal{C}} J(K)$$



Opportunities

- **Easy-to-implement**
- **Scalable** to high-dimensional problems
- Enable **model-free search** with rich observations (e.g. images)

Challenges

- **Nonconvex optimization**
- Lack of principled algorithms for **optimality** (e.g., avoiding saddles/local minimizers)
- Hard to obtain **theoretical guarantees** (e.g., robustness/stability, sample efficiency)

Today's topic: Policy optimization

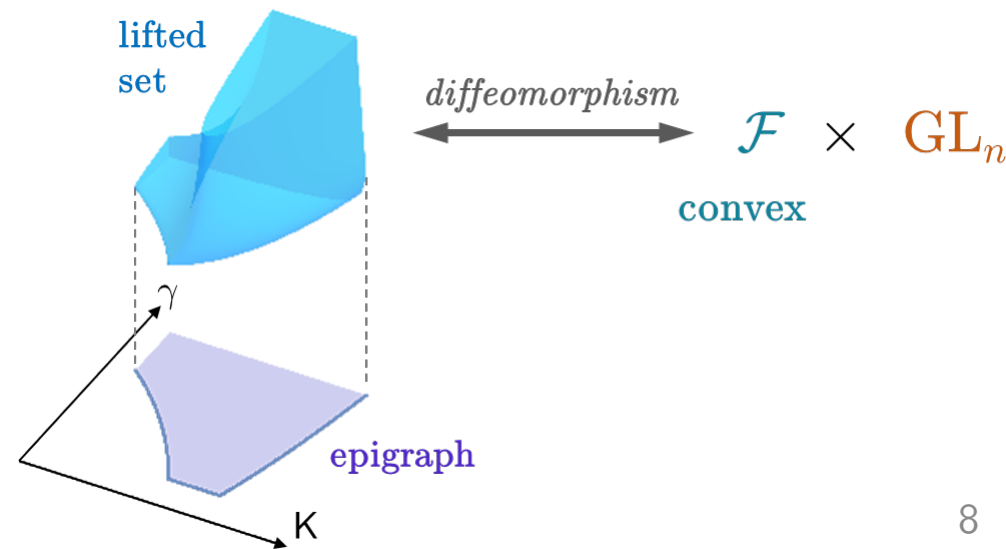
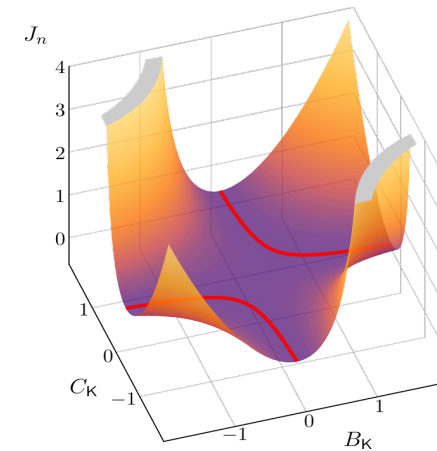
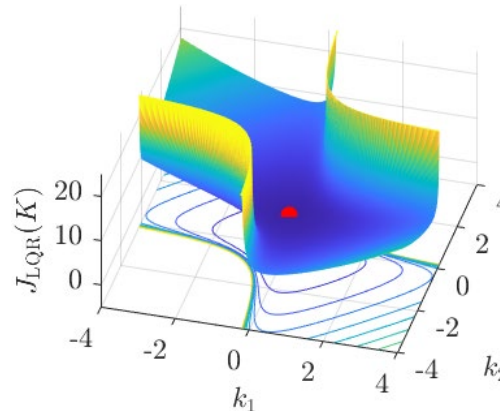
Message 1: Nonconvex Landscapes in Benchmark Control Problems are Benign

- Informally, any (non-degenerate) stationary point is globally optimal!

[Fazel et al., 2018], [Mohammad et al., 2022] [Fatkullin & Polyak, 2021] [Guo & Hu, 2022],
[Tang, Zheng, & Li, 2023], [Zheng, Pai & Tang, 2023] etc.

Message 2: A Unified Extended Convex Lifting Framework

- Bridge non-convex policy optimization with convex reformulation!
 - Zheng, Pai, and Tang. "Benign Nonconvex Landscapes in Optimal and Robust Control, Part II: Extended Convex Lifting." arXiv preprint arXiv:2406.04001



Outline

- **Part I: Benign Nonconvex Landscapes in Control**
- **Part II: Extended Convex Lifting**

Non-convexity in control

Policy optimization in control is generally **nonconvex!**

$$\begin{aligned} \min_K \quad & J(K) \\ \text{s.t.} \quad & K \in \mathcal{C} \end{aligned}$$

$$\min_{K \in \mathbb{R}^{m \times n}} \sum_{t=0}^{\infty} (x_t^\top Q x_t + u_t^\top R u_t) \longrightarrow \text{Quadratic costs}$$

subject to $x_{t+1} = Ax_t + Bu_t$, $\longrightarrow$ **Linear dynamics**

$u_t = Kx_t \longrightarrow$ **Linear static policy**

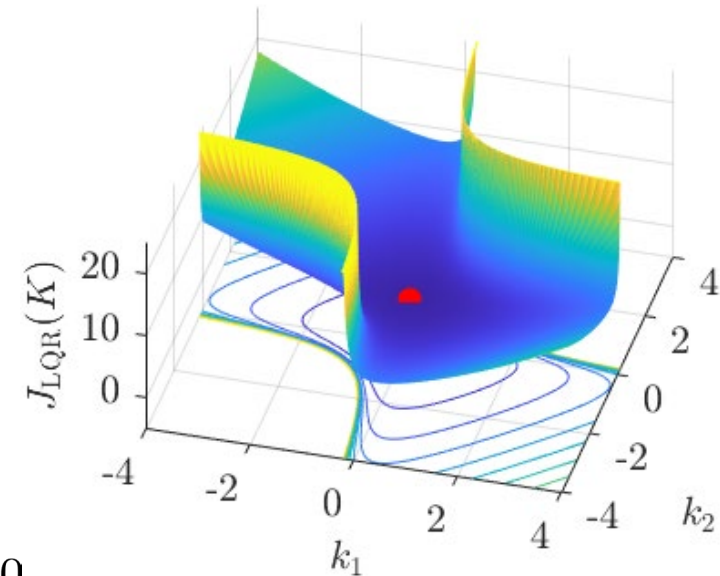
- State trajectories are nonlinear in K

$$x_1 = (A + BK)x_0,$$

$$x_2 = (A + BK)x_1 = (A + BK)^2 x_0$$

$$x_t = (A + BK)^t x_0$$

- The cost function is nonconvex in K



Non-convexity in control

Policy optimization in control is generally **nonconvex**!

- **Stabilization** is already a **non-convex** constraint!

✓ A policy K is said to **stabilize** the system, if the system trajectory converges to the origin.

$$\lim_{t \rightarrow \infty} x_t = \lim_{t \rightarrow \infty} (A + BK)^t x_0 = 0$$

✓ In discrete-time, it requires the spectral radius of the closed-loop matrix less than 1.

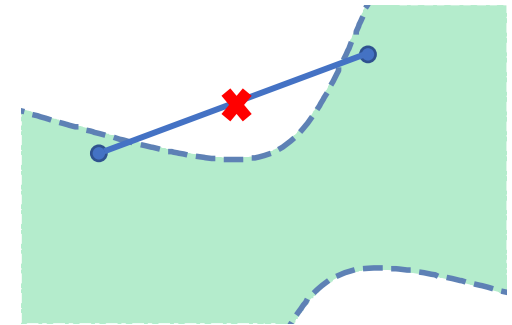
$$\rho(A + BK) < 1$$

✓ A simple example: $A = 0, \quad B = I_2$

$$x_{t+1} = u_t \quad \mathcal{C} = \{K \in \mathbb{R}^{2 \times 2} \mid A + BK \text{ is stable}\}$$

$$\min_K J(K)$$

$$\text{s.t. } K \in \mathcal{C}$$



$$K_1 = \begin{bmatrix} 0.5 & 2 \\ 0 & 0.5 \end{bmatrix} \in \mathcal{C}, \quad K_2 = \begin{bmatrix} 0.5 & 0 \\ 2 & 0.5 \end{bmatrix} \in \mathcal{C},$$

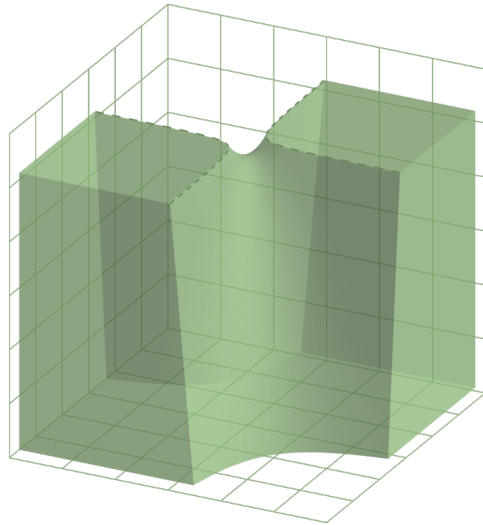
$$\frac{1}{2}(K_1 + K_2) = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix} \notin \mathcal{C}$$

Non-convexity in control

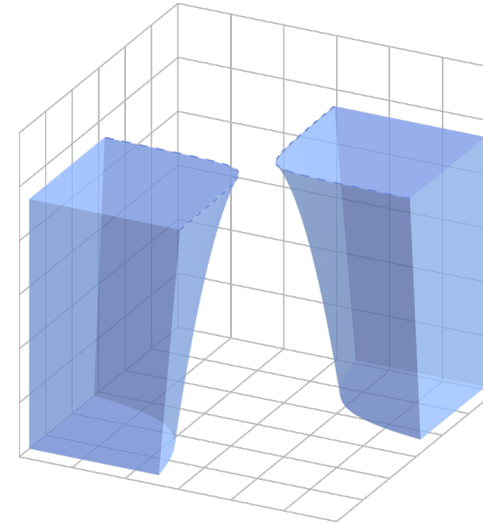
Policy optimization in control is generally **nonconvex**!

$$\begin{aligned} \min_{\mathbf{K}} \quad & J(\mathbf{K}) \\ \text{s.t.} \quad & \mathbf{K} \in \mathcal{C} \end{aligned}$$

- The set of dynamic stabilizing policies is **nonconvex** and may even be **disconnected**. [Tang, Zheng, Li, *Math. Prog.* 2023]



Example 1: Nonconvex but
connected



Example 2: Nonconvex and
dis-connected

Benign non-convexity in control

Policy optimization in **linear** control is generally **nonconvex but benign!**

- Including all iconic benchmark control problems:
 - ✓ LQR, LQG, H2/Hinf optimal control, etc.

□ Linear Quadratic Regulator (LQR)

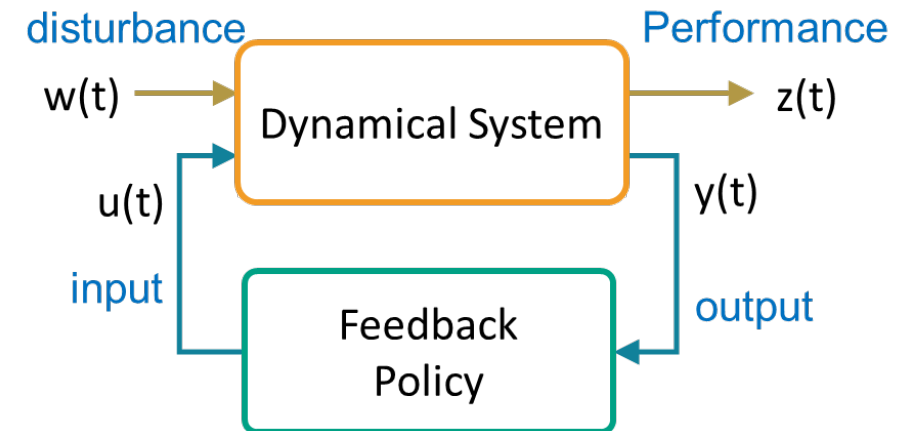
Dynamics: $\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t),$

Static policies: $u(t) = Kx(t)$

Stability: $\mathcal{C} = \{K \in \mathbb{R}^{m \times n} \mid A + BK \text{ is stable}\}$

Performance: $J_{\text{LQR}}(K) := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \int_0^T x^\top(t) Q x(t) + u^\top(t) R u(t) dt \right]$

$$\begin{aligned} \min_K \quad & J(K) \\ \text{s.t.} \quad & K \in \mathcal{C} \end{aligned}$$



Benign non-convexity in LQR

□ Linear Quadratic Regulator (LQR)

Stability: $\mathcal{C} = \{K \in \mathbb{R}^{m \times n} \mid A + BK \text{ is stable}\}$

Fact 1: the set of stabilizing gains is always **path-connected**

Proof: **convexification** via a **nonlinear change of variables**

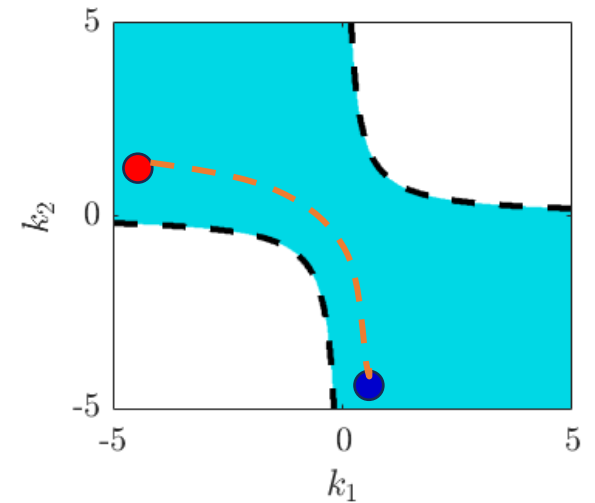
$$\{K \in \mathbb{R}^{m \times n} \mid A + BK \text{ is stable}\}$$

$$\iff \{K \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, P(A + BK)^\top + (A + BK)P \prec 0\}$$

$$\iff \{K \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, PA^\top + L^\top B^\top + AP + BL \prec 0, L = KP\}$$

$$\iff \{K = LP^{-1} \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, PA^\top + L^\top B^\top + AP + BL \prec 0\}.$$

$$\begin{aligned} \min_K & J(K) \\ \text{s.t. } & K \in \mathcal{C} \end{aligned}$$



Any convex set is path-connected, and thus so is its continuous image.

Benign non-convexity in LQR

□ Linear Quadratic Regulator (LQR)

$$J_{\text{LQR}}(K) := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \int_0^T x^\top(t) Q x(t) + u^\top(t) R u(t) dt \right]$$

$$\begin{aligned} \min_K \quad & J(K) \\ \text{s.t.} \quad & K \in \mathcal{C} \end{aligned}$$

Fact 2: the LQR cost is real analytical and thus **infinitely differentiable** over $\mathcal{C}$.

✓ The LQR cost is actually a **rational function** in K

✓ Another form $J_{\text{LQR}}(K) = \text{Tr} [(Q + K^\top R K) X]$

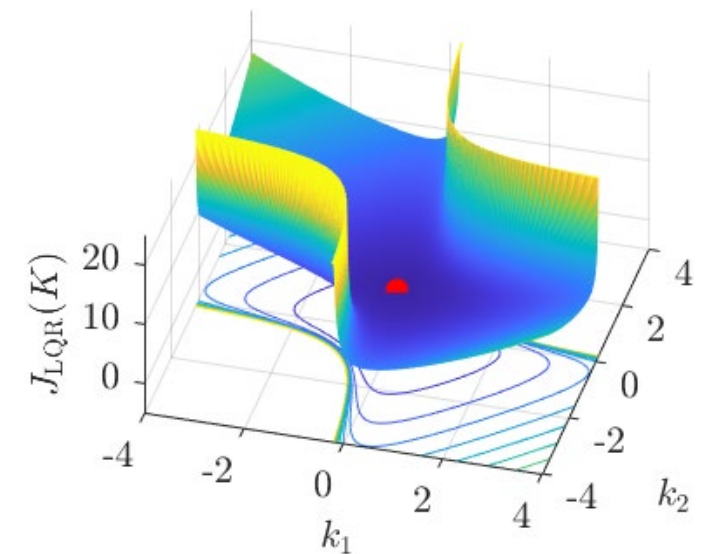
Lyapunov equation $(A + BK)X + X(A + BK)^\top + W = 0$

Fact 3: **Any stationary point** $\nabla J(K) = 0$ **is globally optimal**

Local
Stationarity



Global
Optimality



Benign non-convexity in LQR

□ Linear Quadratic Regulator (LQR)

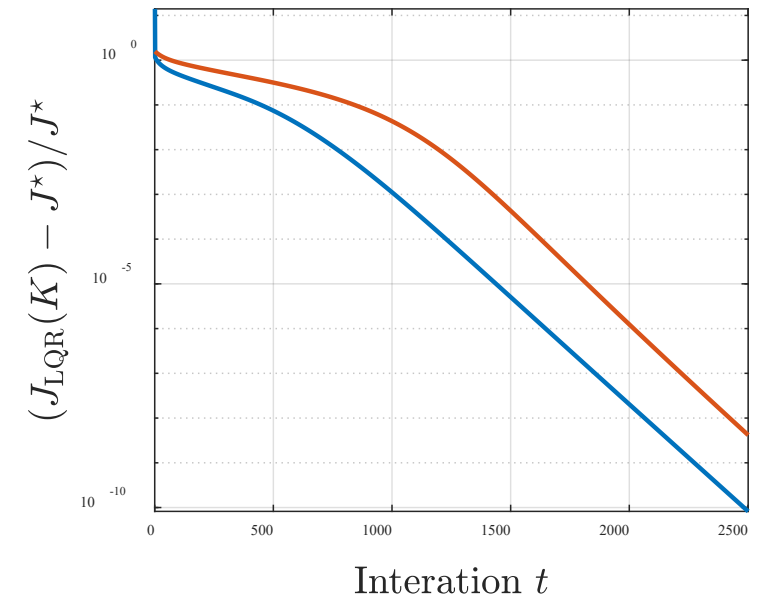
Fact 4: the LQR cost satisfies the **gradient dominance** property over any sublevel set (Fazel et al., 2019; Mohammadi et al., 2022).

$$\begin{aligned} \min_K \quad & J(K) \\ \text{s.t.} \quad & K \in \mathcal{C} \end{aligned}$$

$$J_{\text{LQR}}(K) - J^* \leq \mu \|\nabla J_{\text{LQR}}(K)\|_F^2, \quad \forall K \in [J_{\text{LQR}} \leq \nu]$$

- LQR almost behaves like a **strongly convex** problem under mild assumptions (Watanabe and Zheng, 2025).
- The basic gradient descent algorithm achieves linear convergence (Fazel et al., 2019; Mohammad et al., 2022; Fatkhullin & Polyak, 2021)

$$K_{t+1} = K_t - \alpha \nabla J(K_t)$$



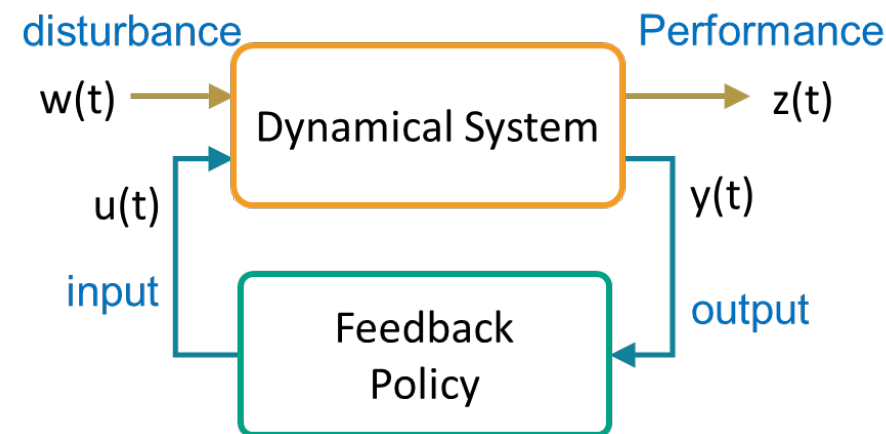
Benign non-convexity in robust control

□ State-feedback H_{∞} robust control

Dynamics: $\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t),$

Static policies: $u(t) = Kx(t)$

Stability: $\mathcal{C} = \{K \in \mathbb{R}^{m \times n} \mid A + BK \text{ is stable}\}$



Unlike LQR with stochastic noises, we here assume **adversarial noises** with bounded energy and consider the **worst-case performance**

Performance: $J_{\infty}(K) := \sup_{\|w(t)\|_2 \leq 1} \int_0^{\infty} x^{\top}(t)Qx(t) + u^{\top}(t)Ru(t) dt$

Benign non-convexity in robust control

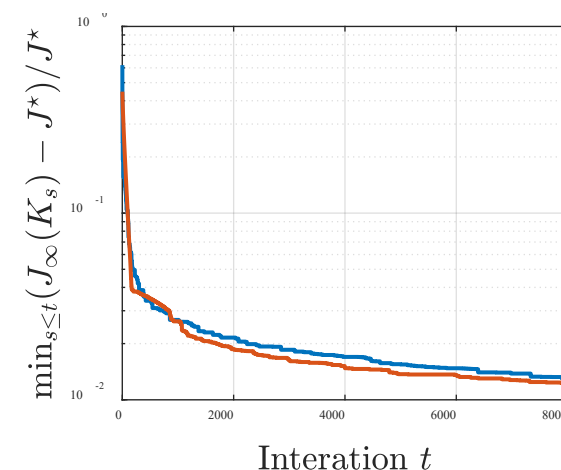
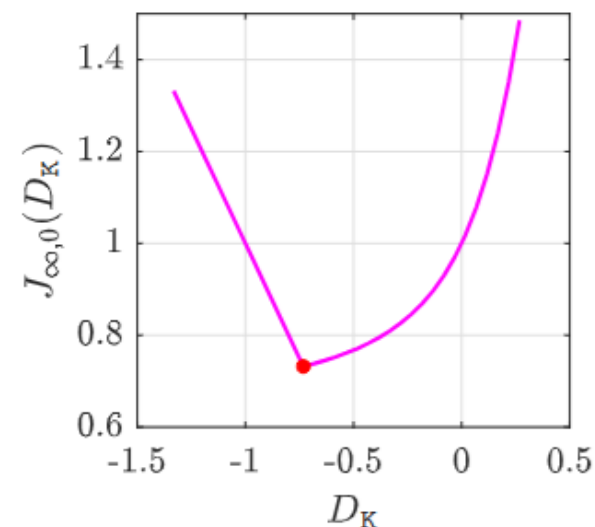
□ State-feedback Hinf robust control

Performance: $J_\infty(K) := \sup_{\|w(t)\|_2 \leq 1} \int_0^\infty x^\top(t) Q x(t) + u^\top(t) R u(t) dt$

$$\begin{aligned} \min_K \quad & J(K) \\ \text{s.t.} \quad & K \in \mathcal{C} \end{aligned}$$

Fact 1: The Hinf cost is generally **non-convex**, and **non-smooth**, but **locally weakly convex**.

Fact 2: **Any sublevel set is path-connected**, and **any stationary point is globally optimal** (Guo and Hu, 2022; Zheng et al., 2023)

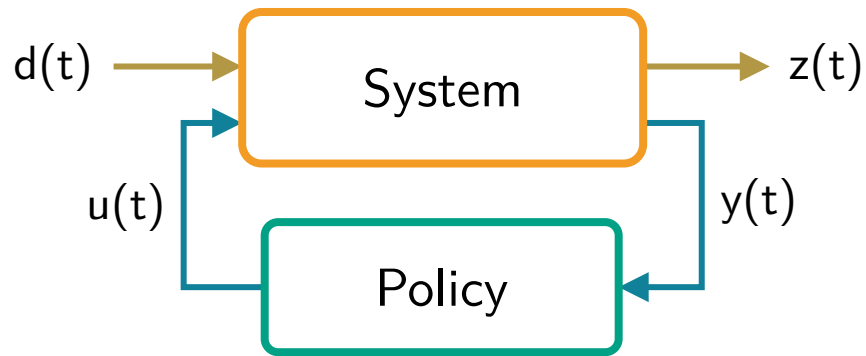


➤ We may use the subgradient method to find a global minimizer despite the non-convexity (Watanabe, Liao, & Zheng, 2025)

$$\begin{aligned} K_{t+1} &= K_t - \alpha_t g_t, \\ g_t &\in \partial J_\infty(K_t) \end{aligned}$$

Benign non-convexity in LQG

□ Linear Quadratic Gaussian (LQG) Control



Dynamics: $\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t)$

$$y(t) = Cx(t) + D_v v(t)$$

Performance:

$$J_{\text{LQG}} := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \int_0^T x^\top(t) Q x(t) + u^\top(t) R u(t) dt \right]$$

Separation principle:

Kalman filter for state estimation
+ LQR based on the estimated state

$$\begin{aligned} \dot{\xi} &= (A - BK)\xi + L(y - C\xi), \\ u &= -K\xi. \end{aligned}$$

Policy parameterization

➤ We need to consider **dynamic policies**:

$$\begin{aligned} \dot{\xi}(t) &= A_K \xi(t) + B_K y(t) \\ u(t) &= C_K \xi(t) \end{aligned}$$

$$K = (A_K, B_K, C_K) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times p} \times \mathbb{R}^{m \times n}$$

We say this is a **full-order** policy

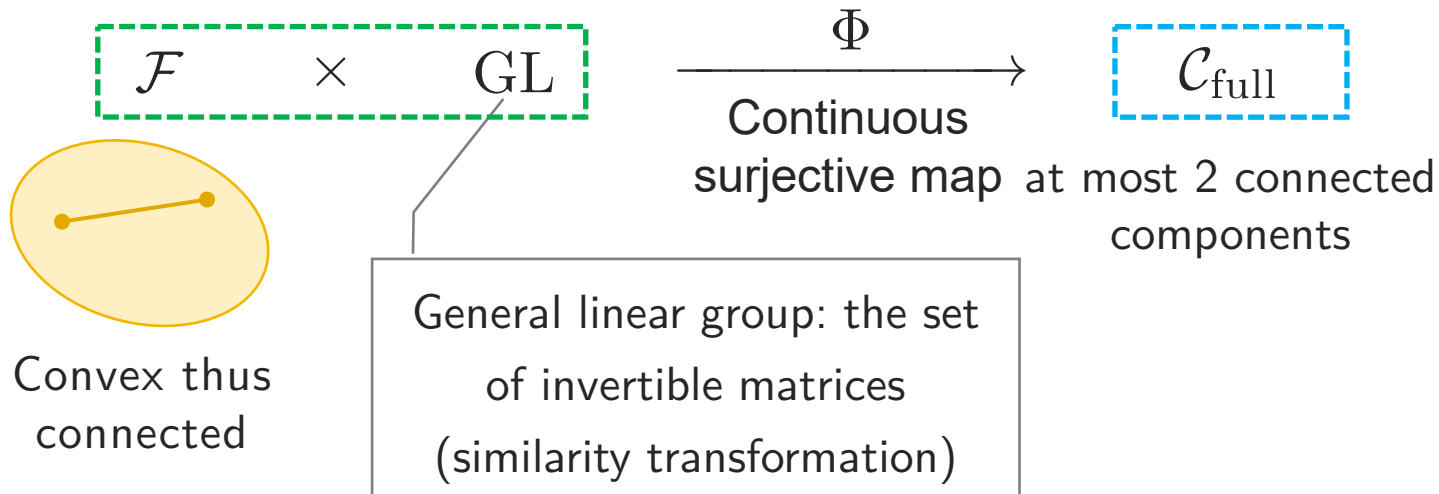
Benign nonconvexity in LQG

Linear Quadratic Gaussian (LQG) Control

$$\mathcal{C}_{\text{full}} = \left\{ K \mid K = (A_K, B_K, C_K) \text{ is full order} \right. \\ \left. \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \text{ is Hurwitz stable} \right\}$$

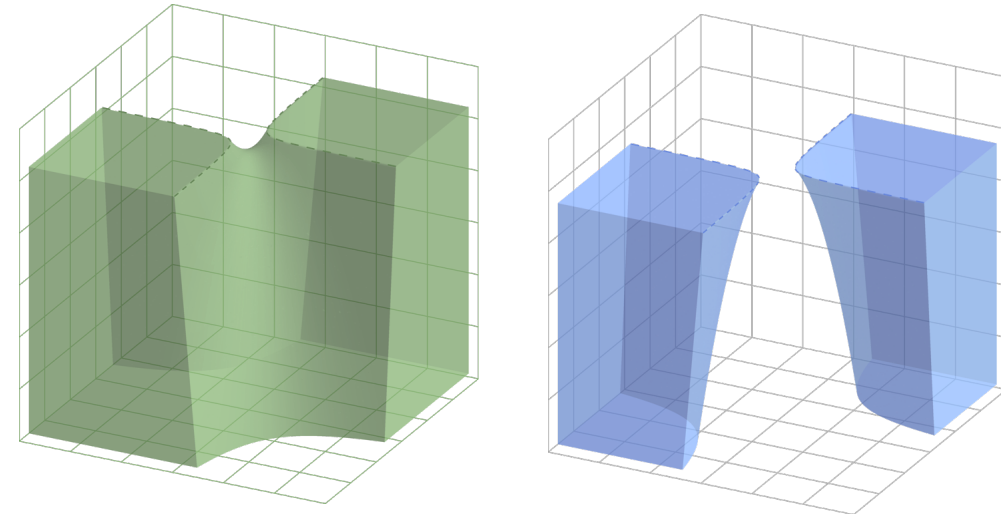
Fact 1: The set $\mathcal{C}_{\text{full}}$ can be **disconnected** but has at most **2 connected** components [Tang, Zheng, Li, 2023].

Proof: **convexification** via a **nonlinear change of variables**



Policy optimization for LQG

$$\min_K J(K) \\ \text{s.t. } K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$$



Two connected components

$$\text{GL}_n^+ = \{ \Pi \in \mathbb{R}^{n \times n} \mid \det \Pi > 0 \}, \\ \text{GL}_n^- = \{ \Pi \in \mathbb{R}^{n \times n} \mid \det \Pi < 0 \}.$$

Benign nonconvexity in LQG

□ Linear Quadratic Gaussian (LQG) Control

Fact 2: For a **controllable and observable** policy, if it is a **stationary** point $\nabla J(K) = 0$, then it is a **global minimizer** [Tang, Zheng, Li, 2023].

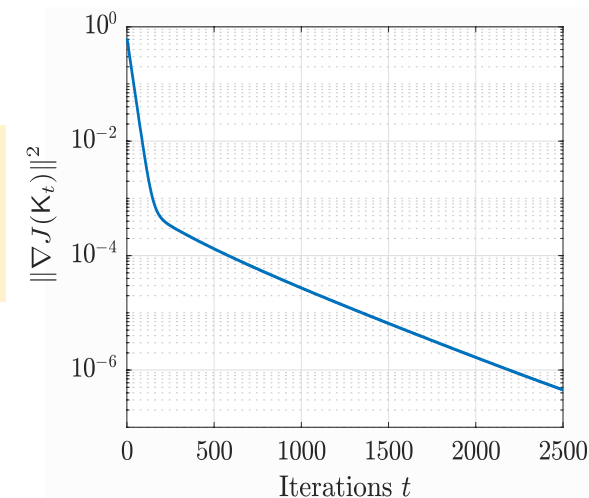
Local Zero
Gradient



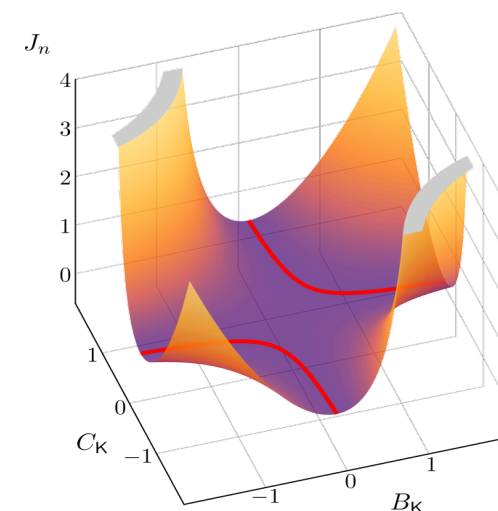
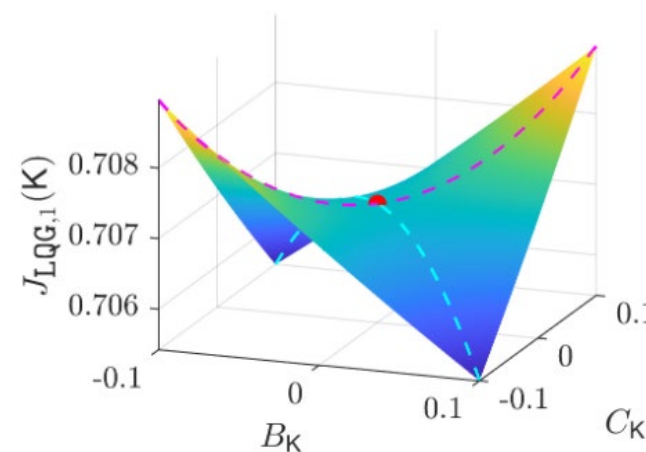
Structural
Information



Global
Optimality
Certificate



Fact 3: For policies that are uncontrollable or unobservable, there exists suboptimal stationary points (e.g., **saddle points**) [Tang, Zheng, Li, 2023].



Benign Nonconvex Policy Optimization

□ Recent progress on non-convex policy optimization

- **Favorable properties** have been revealed for policy optimization in many benchmark control problems:
 - ✓ LQR [Fazel et al., 2018] [Mohammad et al., 2022] [Fatkhullin & Polyak, 2021] [Watanabe & Zheng, 2025], etc.
 - ✓ LQG [Tang, Zheng, & Li, 2021], [Zheng et al., 2022], [Ren et al., 2023] [Kraisler & Mesbahi, 2024], etc.
 - ✓ $\mathcal{H}_\infty$ state-feedback/output-feedback, [Guo & Hu, 2022] [Hu & Zheng, 2022] [Zheng et al, 2023 & 2024]

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Review Article | Open Access

Toward a Theoretical Foundation for Learning Control Policies

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Policy Optimization in Control: Geometry and Algorithmic Implications

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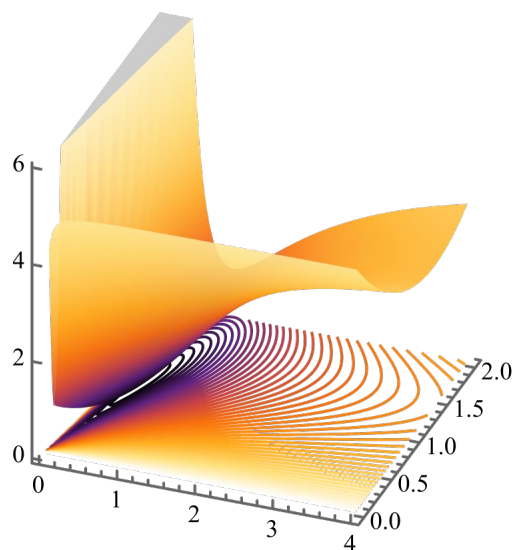
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Outline

- Part I: Benign Nonconvex Landscapes in Control
- **Part II: Extended Convex Lifting**

Key idea

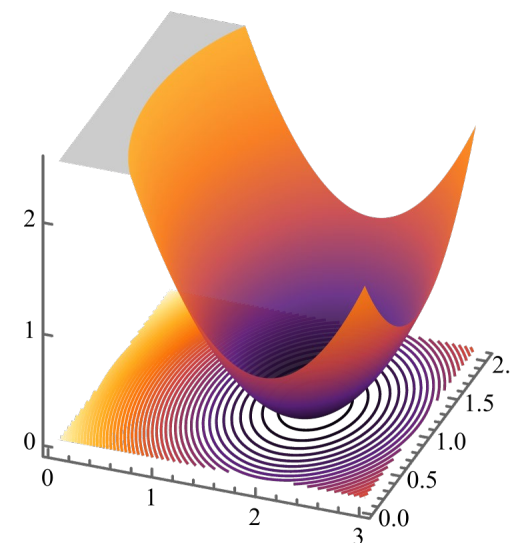
Benign Non-convexity in Control via Extended Convex Lifting (ECL)



Nonconvex
policy
optimization



LMI-based
convex
reformulation

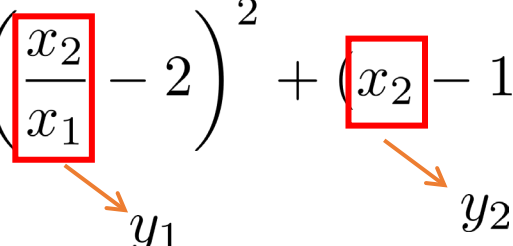


- Our idea: Exploit **LMI-based convex reformulations** of control problems
- They reveal the **hidden convexity** of policy optimization landscapes.
- For **non-degenerate** policies, **all Clarke stationary points are globally optimal** and there is **no spurious local minimum** in policy optimization.

Example 1

□ Nonconvex and smooth function

$$f_1(x_1, x_2) = \left(\frac{x_2}{x_1} - 2 \right)^2 + (x_2 - 1)^2, \quad \text{dom}(f_1) = \{x \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}.$$

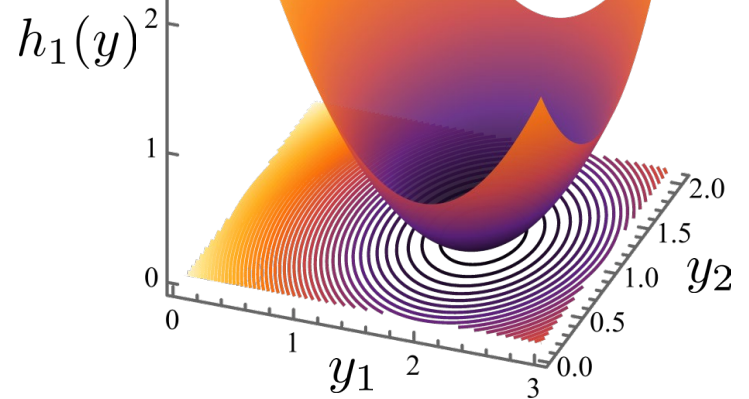


Its global **minimizer** is

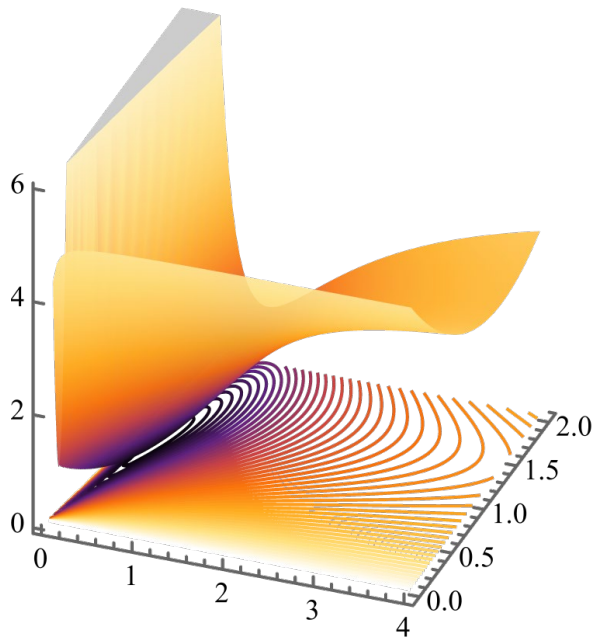
$$x^* = (0.5, 1)$$

Define an **invertible map**

$$g(x) := (x_2/x_1, x_2), \\ \forall x_1 > 0, x_2 > 0,$$



$$h_1(y) := f_1(g^{-1}(y)) = (y_1 - 2)^2 + (y_2 - 1)^2, \quad \forall y_1 > 0, y_2 > 0.$$



Example 2

□ Nonconvex and non-smooth function

$$f_2(x_1, x_2) = \left| \frac{x_2}{x_1} - 2 \right| + |x_2 - 1|, \quad \text{dom}(f_2) = \{x \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}.$$

$\xrightarrow{\quad y_1 \quad}$ $\xrightarrow{\quad y_2 \quad}$

Its global **minimizer** is

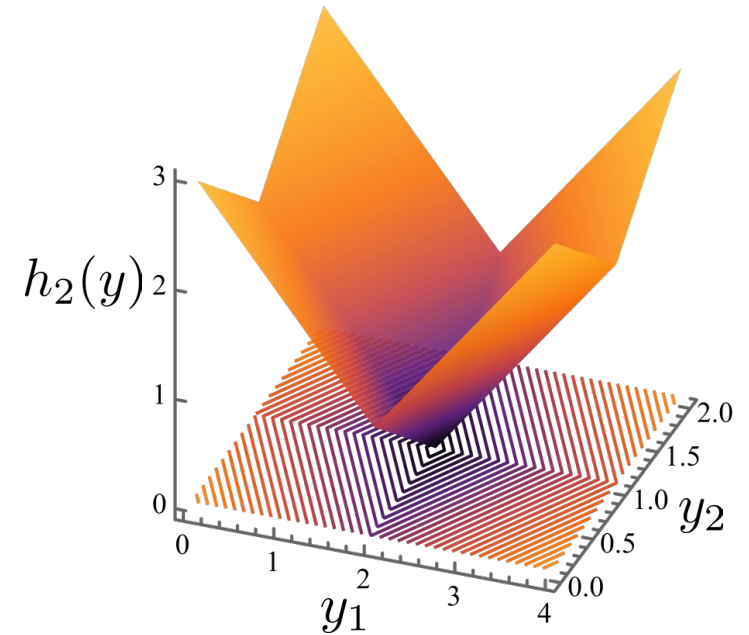
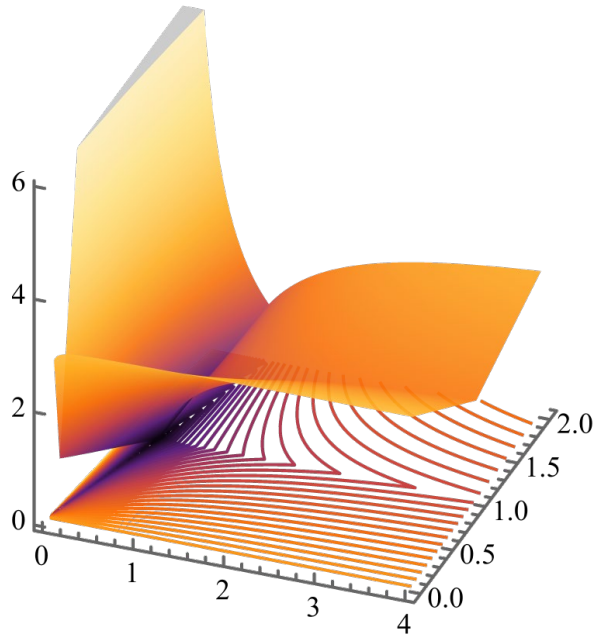
$$x^* = (0.5, 1)$$

Define an **invertible map**

$$g(x) := (x_2/x_1, x_2), \\ \forall x_1 > 0, x_2 > 0,$$



$$h_2(y) := f_2(g^{-1}(y)) = |y_1 - 2| + |y_2 - 1|, \quad \forall y_1 > 0, y_2 > 0,$$



Example 3

□ Linear Quadratic Regulator (LQR)

$$J(k_1, k_2) = \frac{1 - 2k_2 + 3k_2^2 - 2k_2^3 - 2k_1^2 k_2}{k_2^2 - 1}, \quad \forall k_1 \in \mathbb{R}, k_2 < -1.$$

- Not easy to see whether it is convex in the current form
- This cost function comes from an **LQR instance**

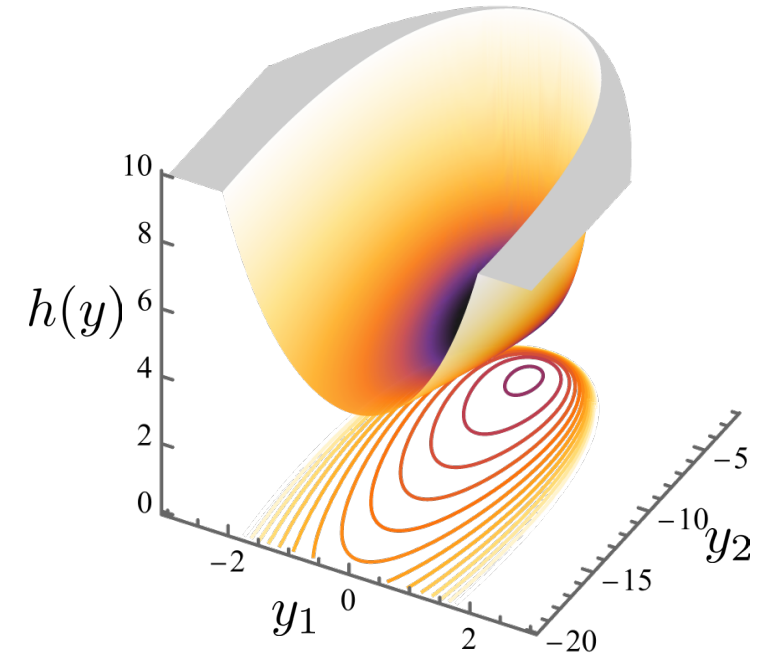
$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = I_2, R = 1$$

- There exists an **invertible mapping**

$$g(k) := \left(\frac{k_1}{1 - k_2}, \frac{2k_2 - k_1^2 - 2k_2^2}{k_2^2 - 1} \right) \quad \forall k_1 \in \mathbb{R}, k_2 < -1.$$

- We get a **convex function** in terms of the new variable y

$$h(y) := J(g^{-1}(y)) = -y_2 - 1 + y^\top \begin{bmatrix} 1 & y_1 \\ y_1 & -y_2 - 2 \end{bmatrix}^{-1} y, \quad \forall \begin{bmatrix} 1 & y_1 \\ y_1 & -y_2 - 2 \end{bmatrix} \succ 0.$$



Direct Convex Reformulation

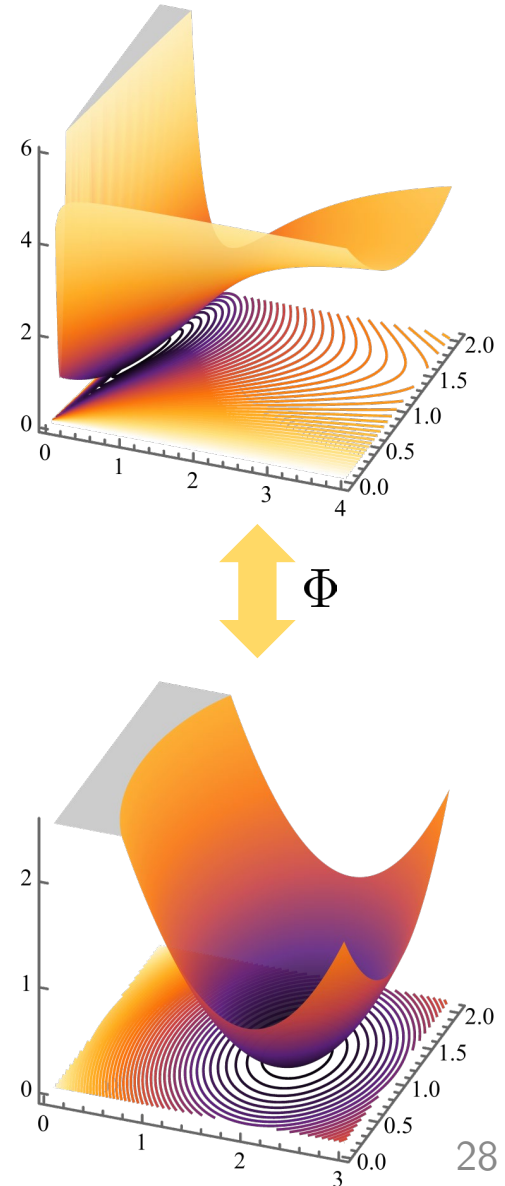
□ Direct convex reformulation (no lifting)

- Consider a continuous function $J(K) : \mathcal{D} \rightarrow \mathbb{R}$.
Denote its epigraph as $\text{epi}_{\geq}(f) := \{(K, \gamma) \in \mathcal{D} \times \mathbb{R} \mid \gamma \geq J(K)\}$.
- Suppose there exists a **smooth and invertible map** Φ between
 $\text{epi}_{\geq}(J)$ and a **convex set** $\mathcal{F}_{\text{cvx}}$
- and we further have $(y, \gamma) = \Phi(K, \gamma), \forall (K, \gamma) \in \text{epi}_{\geq}(J)$

Guarantee 1: Minimizing J is equivalent to a convex problem

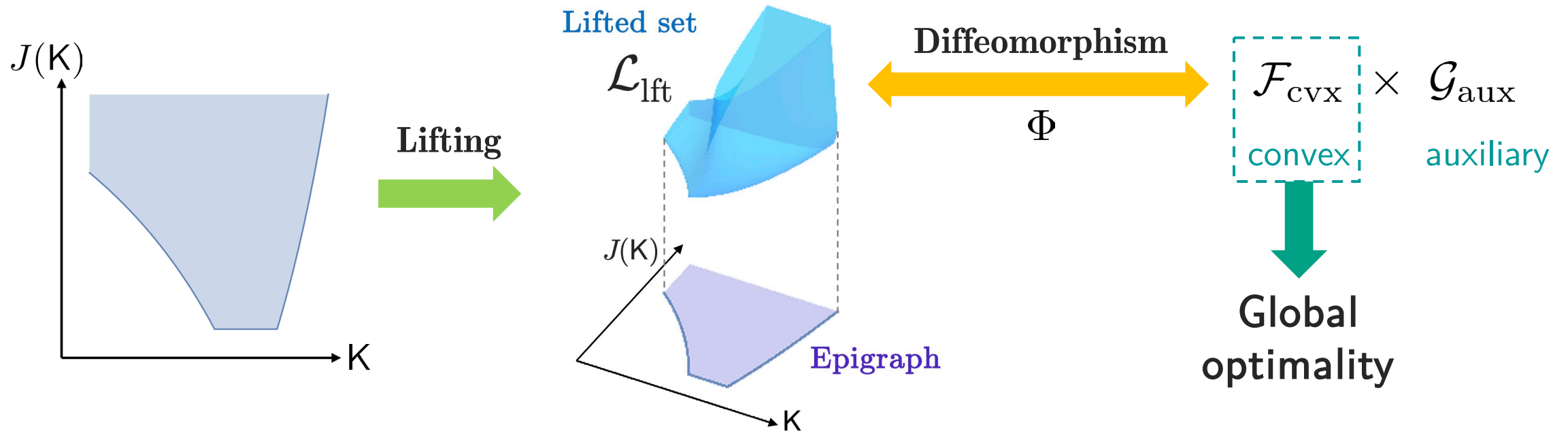
$$\inf_{K \in \mathcal{D}} J(K) = \inf_{(y, \gamma) \in \mathcal{F}_{\text{cvx}}} \gamma.$$

Guarantee 2: Any stationary point to J is globally optimal; in other words, $0 \in \partial J(K^*)$ implies globally optimality



Extended Convex Lifting (ECL)

A schematic illustration of ECL:



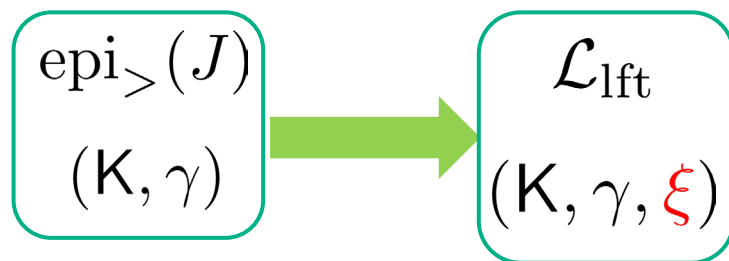
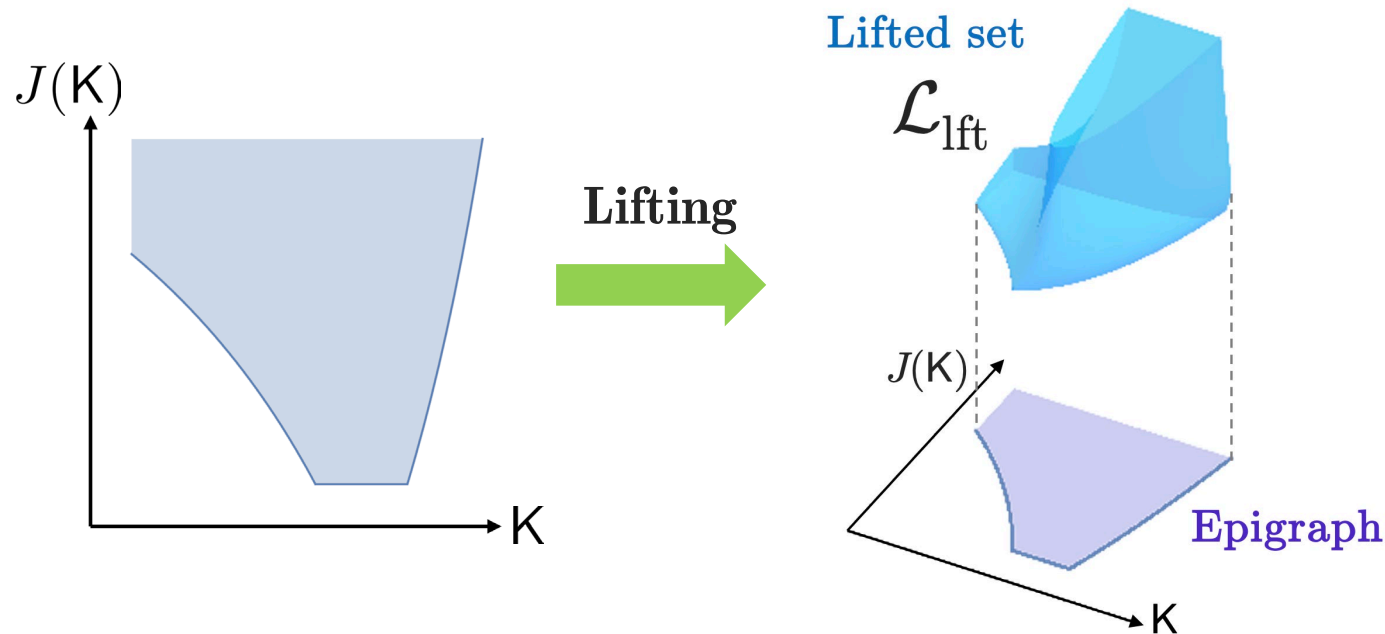
Two key features

Feature 1: a lifting procedure

Feature 2: an auxiliary set

Extended Convex Lifting (ECL)

A schematic illustration of ECL:



Why lifting?

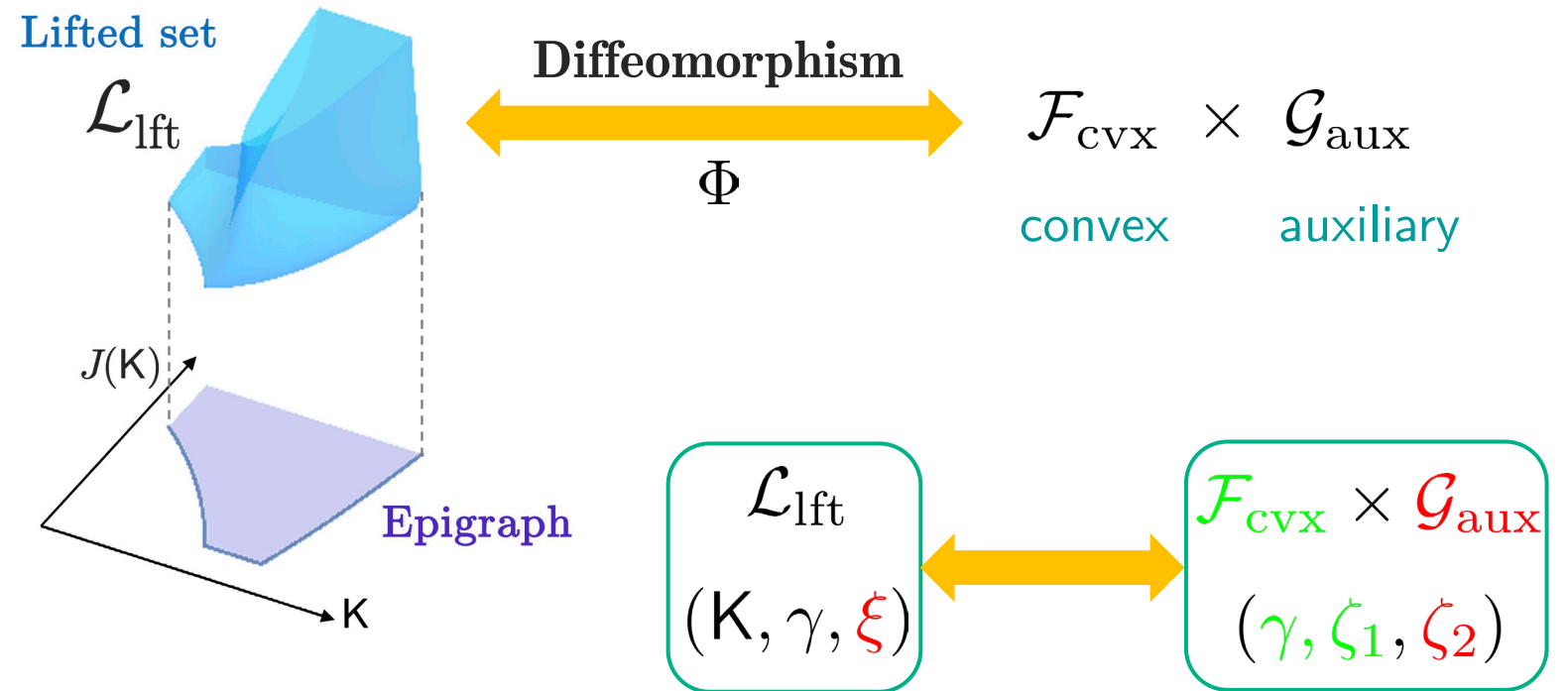
- For many control problems, a **direct convexification is not possible**
- A **lifting procedure** corresponding to **Lyapunov variables** is necessary.

Extended Convex Lifting (ECL)

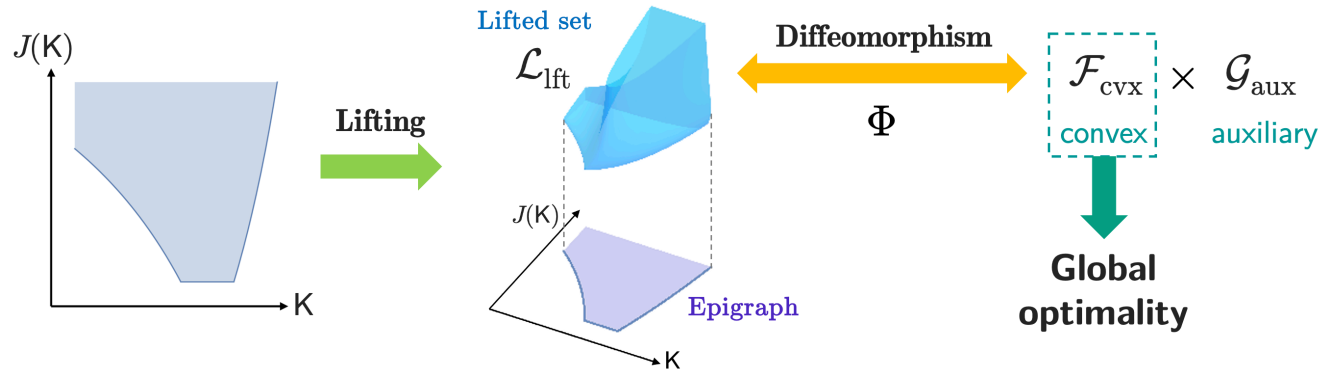
A schematic illustration of ECL:

Why auxiliary set?

- Allows us to isolate the **redundancy** or **symmetry** in the original nonconvex domain
- Related to **similarity transformations** of dynamic policies in control
- Needed for **output-feedback control** problems



Formal ECL Definition



Extended Convex Lifting (ECL)

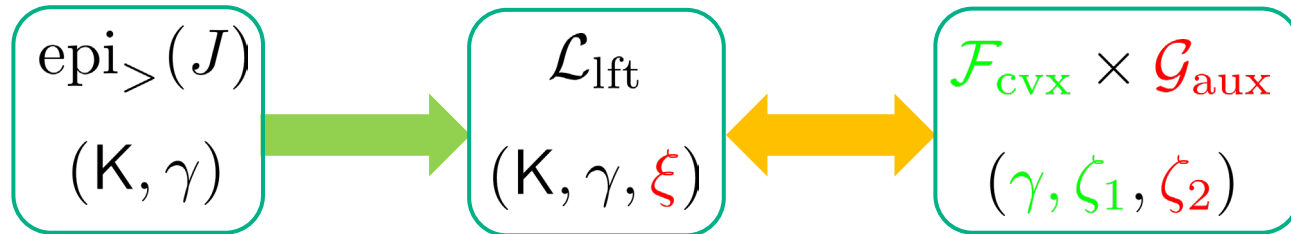
We say a tuple $(\mathcal{L}_{\text{lift}}, \mathcal{F}_{\text{cvx}}, \mathcal{G}_{\text{aux}}, \Phi)$ is an ECL of $J(K) : \mathcal{D} \rightarrow \mathbb{R}$ if

- A lifted set $\mathcal{L}_{\text{lift}}$ satisfying

$$\text{epi}_{>}(J) \subseteq \pi_{K, \gamma}(\mathcal{L}_{\text{lift}}) \subseteq \text{cl epi}_{\geq}(J)$$

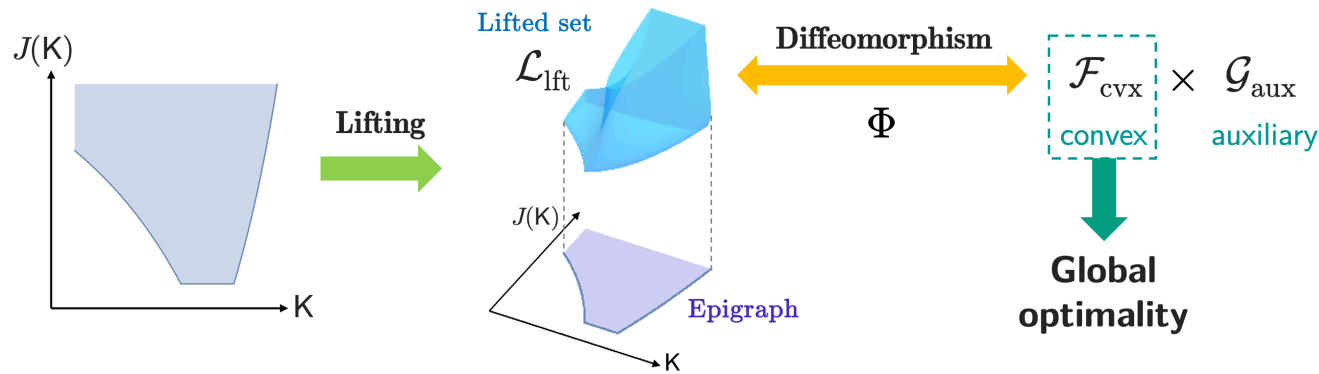
- A diffeomorphism $\Phi : \mathcal{L}_{\text{lift}} \rightarrow \mathcal{F}_{\text{cvx}} \times \mathcal{G}_{\text{aux}}$ such that

$$\Phi(K, \gamma, \xi) = (\gamma, \zeta_1, \zeta_2)$$



- Consider a continuous function $J(K) : \mathcal{D} \rightarrow \mathbb{R}$ where $\mathcal{D} \subseteq \mathbb{R}^d$.
- Denote its strict and non-strict epigraph as $\text{epi}_{>}(J) := \{(K, \gamma) \in \mathcal{D} \times \mathbb{R} \mid \gamma > J(K)\}$, $\text{epi}_{\geq}(J) := \{(K, \gamma) \in \mathcal{D} \times \mathbb{R} \mid \gamma \geq J(K)\}$.

A special ECL



A more intuitive condition

- A lifted set $\mathcal{L}_{\text{lift}}$ satisfying $\text{epi}_{>}(J) \subseteq \pi_{K,\gamma}(\mathcal{L}_{\text{lift}}) \subseteq \text{cl epi}_{\geq}(J)$

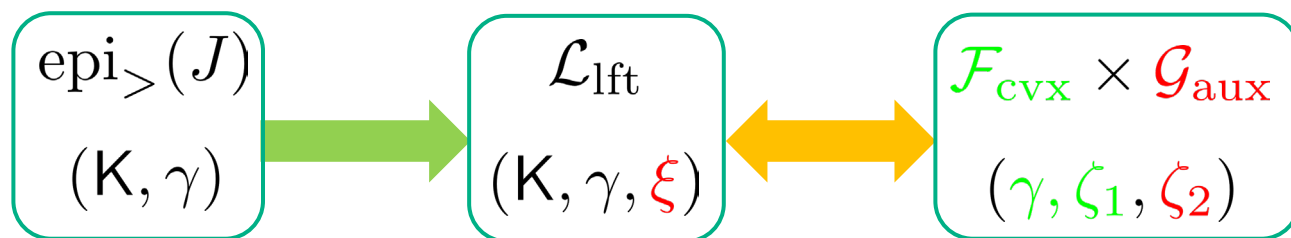
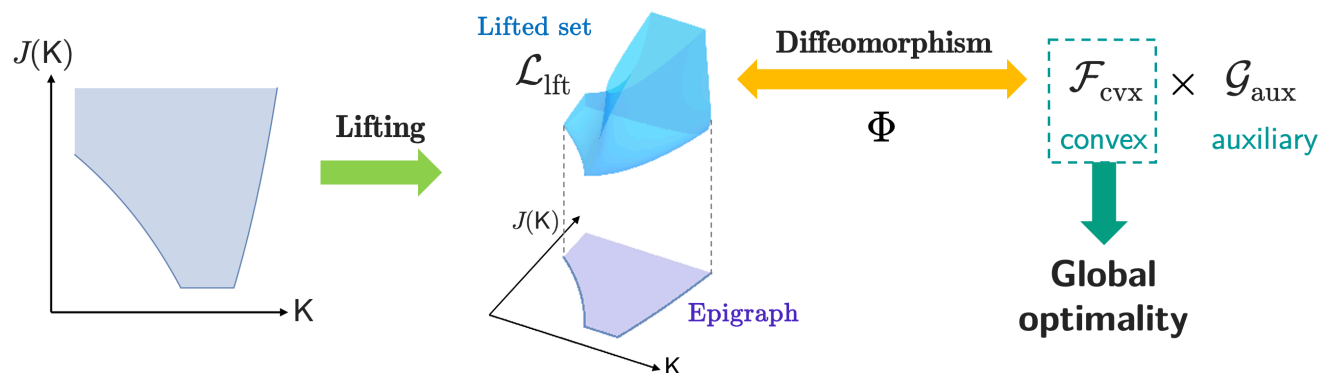


- A lifted set $\mathcal{L}_{\text{lift}}$ satisfying $\pi_{K,\gamma}(\mathcal{L}_{\text{lift}}) = \text{epi}_{\geq}(f)$.

Does this “simpler” lifting condition work?

- Apparently, the condition on the left is **less restrictive**, and works for more general situations
- The simpler condition on the right is indeed sufficient for **state-feedback control** problems
- However, it is too restrictive for **dynamic output-feedback control** problems (e.g., LQG)

Non-degenerate points



Extended Convex Lifting:

- A lifted set $\mathcal{L}_{\text{lift}}$ satisfying

$$\text{epi}_{>}(J) \subseteq \pi_{K,\gamma}(\mathcal{L}_{\text{lift}}) \subseteq \text{cl epi}_{\geq}(J)$$
- A diffeomorphism $\Phi : \mathcal{L}_{\text{lift}} \rightarrow \mathcal{F}_{\text{cvx}} \times \mathcal{G}_{\text{aux}}$ such that

$$\Phi(K, \gamma, \xi) = (\gamma, \zeta_1, \zeta_2)$$

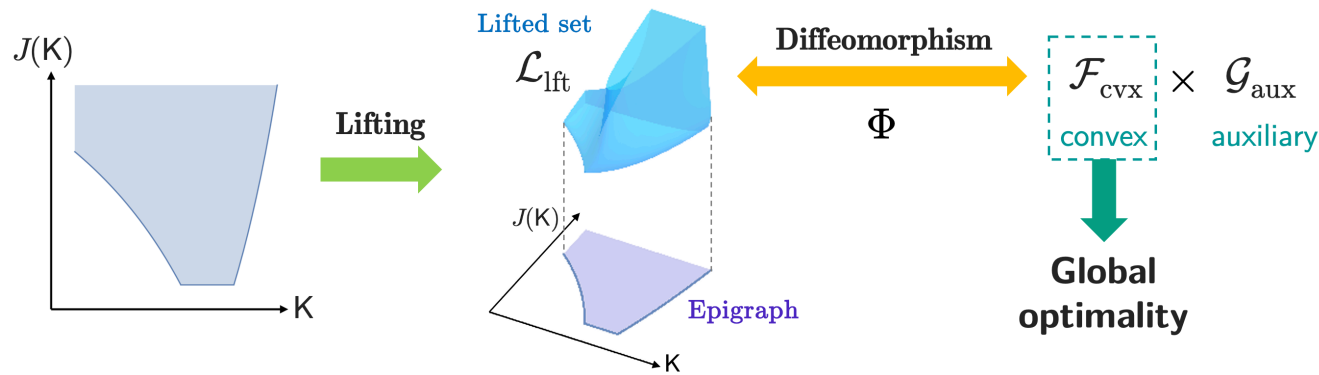
- By construction, some points in $\text{epi}_{\geq}(J)$ may **not be covered** in the lifted set

➡ Those points will be called **degenerate** ➡ bad behavior may happen (e.g., **saddles**)

Definition. K is called **non-degenerate** if $(K, J(K)) \in \pi_{K,\gamma}(\mathcal{L}_{\text{lift}})$

➡ **well-behaved**

ECL Guarantees



Guarantee 1: Convex Reformulation

Optimization $\min_{K \in \mathcal{D}} J(K)$ is
equivalent to a convex
problem

$$\inf_{K \in \mathcal{D}} J(K) = \inf_{(y, \gamma) \in \mathcal{F}_{\text{cvx}}} \gamma.$$

Guarantee 2: Global Optimality

All **non-degenerate** Clarke
stationary points are
globally optimal (for
subdifferentially regular functions)

- Clarke stationary points: Generalization of stationary points to **nonsmooth functions**, based on the notion of **Clarke subdifferential**

Part II: Extended Convex Lifting

1. The ECL framework

2. Applications in control

Linear Quadratic Regulator (LQR)

❑ Problem setup

Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t),$$

Static policies:

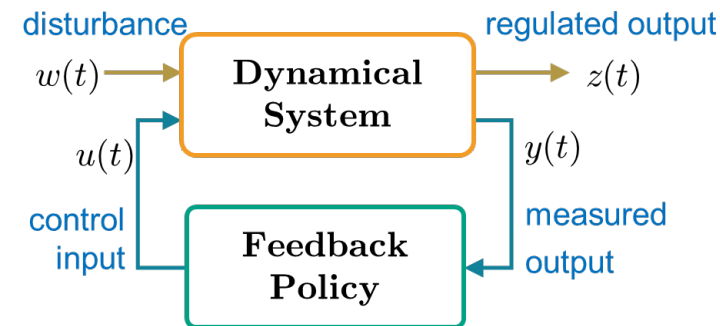
$$u(t) = Kx(t)$$

Stability:

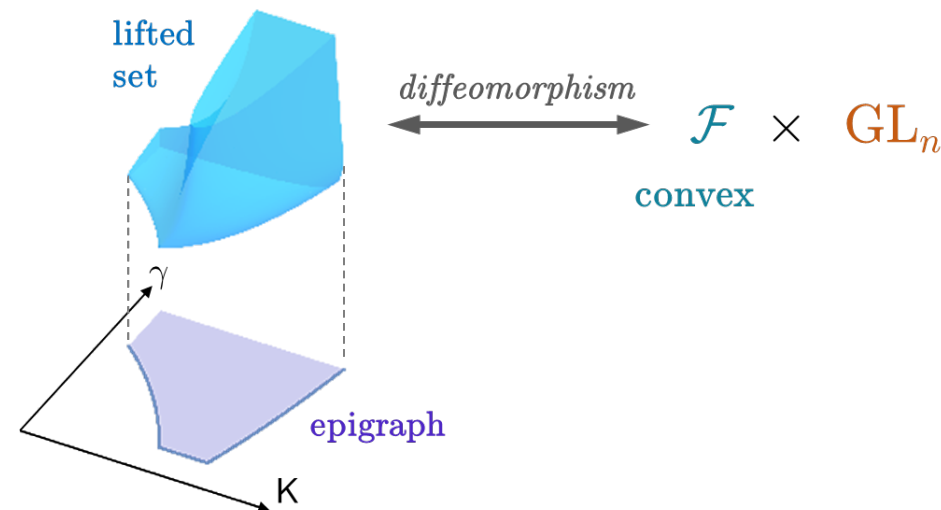
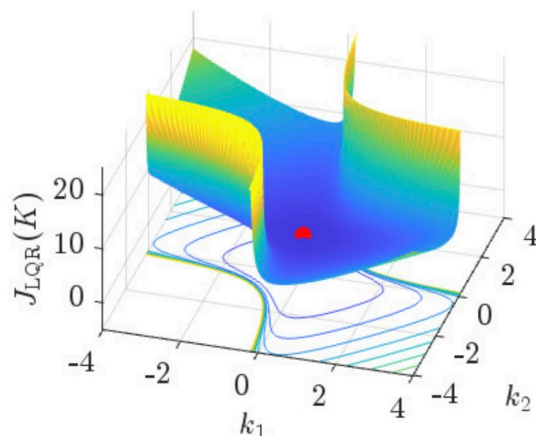
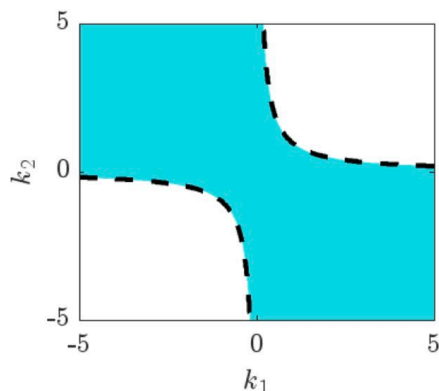
$$\mathcal{C} = \{K \in \mathbb{R}^{m \times n} \mid A + BK \text{ is stable}\}$$

Performance:

$$J_{\text{LQR}}(K) := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \int_0^T x^\top(t) Q x(t) + u^\top(t) R u(t) dt \right]$$



❑ Nonconvex and smooth landscape



Linear Quadratic Regulator (LQR)

□ Construction of ECL

Step 1: Lifting

$$\mathcal{L}_{\text{LQR}} := \left\{ (K, \gamma, \mathbf{X}) : \mathbf{X} \succ 0, (A + BK)\mathbf{X} + \mathbf{X}(A + BK)^\top + W = 0, \gamma \geq \text{Tr} \left[(Q + K^\top R K) \mathbf{X} \right] \right\}.$$

Step 2: Convex set

$$\mathcal{F}_{\text{LQR}} = \left\{ (\gamma, Y, X) : X \succ 0, AX + BY + XA^\top + Y^\top B^\top + W = 0, \gamma \geq \text{tr}(QX + X^{-1}Y^\top RY) \right\}$$

Step 3: Diffeomorphism $\Phi(K, \gamma, X) = (\gamma, KX, X), \quad \forall (K, \gamma, X) \in \mathcal{L}_{\text{LQR}}$

- No auxiliary set
- Lifted set satisfies $\text{epi}_{\geq}(J) = \pi_{K, \gamma}(\mathcal{L}_{\text{LQR}})$

 All policies are non-degenerate

Theorem. Any stationary point of the LQR cost function is globally optimal.

State-feedback Robust Control

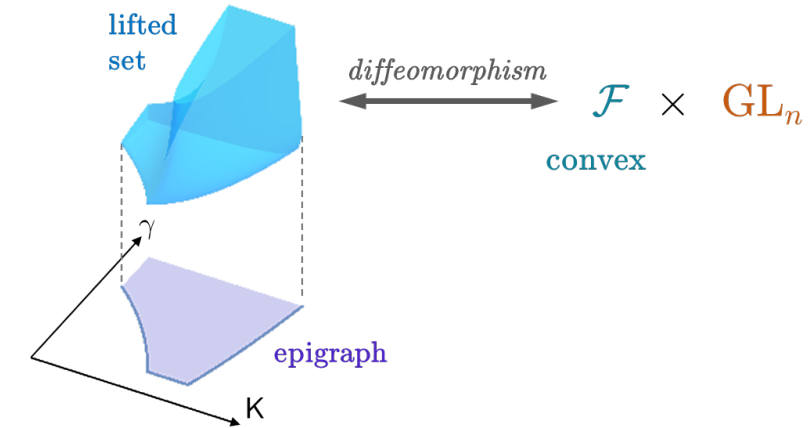
□ Problem setup

Dynamics: $\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t),$

Static policies: $u(t) = Kx(t)$

Stability: $\mathcal{C} = \{K \in \mathbb{R}^{m \times n} \mid A + BK \text{ is stable}\}$

Performance: $J_\infty(K) := \sup_{\|w(t)\|_2 \leq 1} \int_0^\infty x^\top(t) Q x(t) + u^\top(t) R u(t) dt$



□ Building an ECL

Step 1: Lifting

$$\mathcal{L}_\infty := \left\{ (K, \gamma, P) : P \succ 0, \begin{bmatrix} (A + BK)^\top P + P(A + BK) & P B_w & C^\top \\ B_w^\top P & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} \preceq 0 \right\},$$

State-feedback Robust Control

□ Building an ECL

Step 1: Lifting

$$\mathcal{L}_\infty := \left\{ (K, \gamma, \mathbf{P}) : P \succ 0, \begin{bmatrix} (A+BK)^\top \mathbf{P} + \mathbf{P}(A+BK) & \mathbf{P}B_w & C^\top \\ B_w^\top \mathbf{P} & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} \preceq 0 \right\},$$

Step 2: Convex set

$$\mathcal{F}_\infty = \left\{ (\gamma, Y, X) \left| \begin{array}{l} X \succ 0, \\ Y \in \mathbb{R}^{m \times n}, \end{array} \begin{bmatrix} AX + XA^\top + BY + Y^\top B^\top & B_w & XQ^{1/2} & Y^\top R^{1/2} \\ B_w^\top & -\gamma I & 0 & 0 \\ Q^{1/2}X & 0 & -\gamma I & 0 \\ R^{1/2}Y & 0 & 0 & -\gamma I \end{bmatrix} \preceq 0 \right. \right\},$$

Step 3: Diffeomorphism $\Phi(K, \gamma, P) = (\gamma, KP^{-1}, P^{-1}), \quad \forall (K, \gamma, P) \in \mathcal{L}_\infty.$

- No auxiliary set
- Lifted set satisfies $\pi_{K, \gamma}(\mathcal{L}_\infty) = \text{epi}_{\geq}(J_\infty)$

➡ All policies are non-degenerate

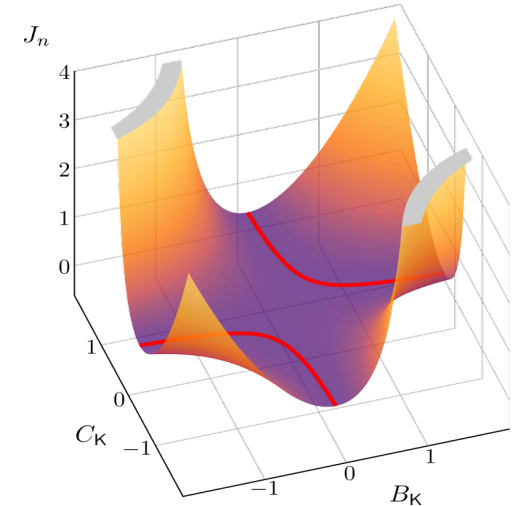
Theorem: Any Clarke stationary points are globally optimal.

Linear Quadratic Gaussian (LQG) Control

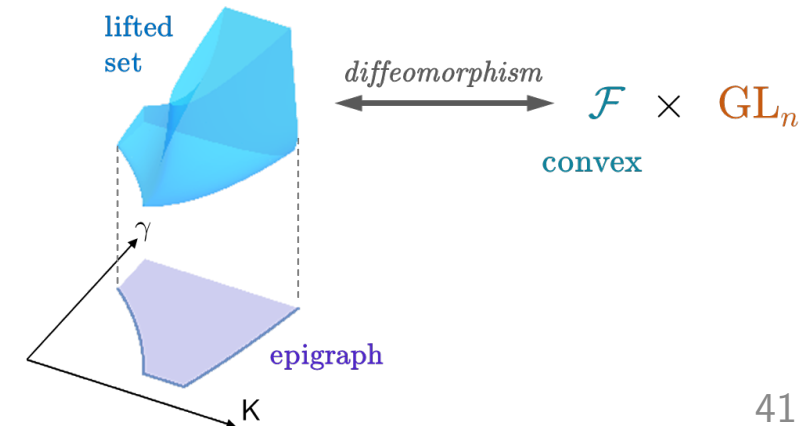
❑ **Construction of the ECL:** Based on the convexification proposed in [Scherer et al., 1997]

Theorem. 1. An ECL for LQG exists, of which the auxiliary set is the set of invertible matrices.

2. A policy K is non-degenerate if and only if it is **informative** in the sense that $\lim_{t \rightarrow \infty} \mathbb{E}[x(t)\xi(t)^T]$ has full rank. So **any informative stationary point is globally optimal**.



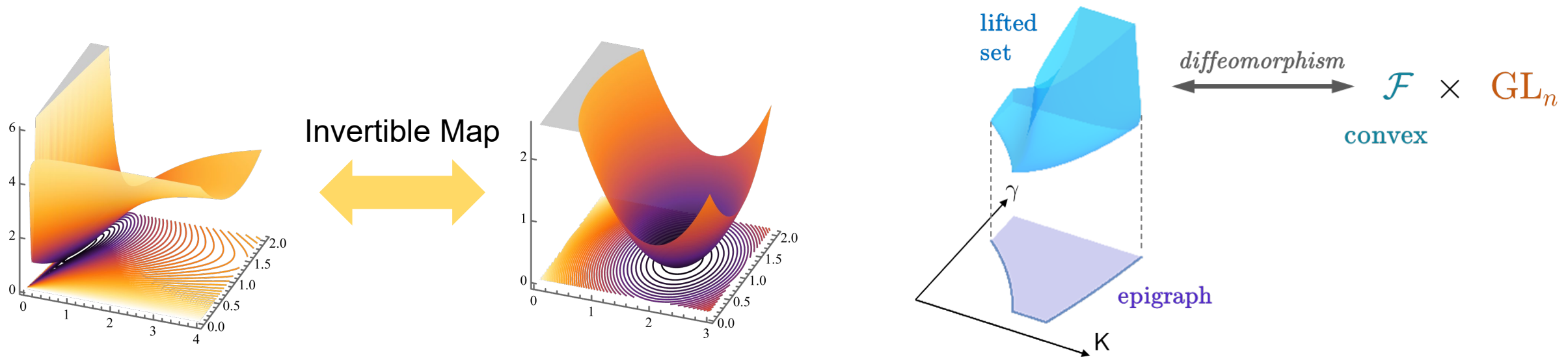
- We also show that **controllable and observable stationary policies are informative**, and are thus globally optimal [Tang, Zheng, Li, 2023].
- Similar ECL guarantees hold for output-feedback Hinf control and other control problems [Zheng, Pai & Tang, 2024].



Conclusion

Nonconvex Policy Optimization in control

- ❑ Policy optimization in control is generally **nonconvex** and **non-smooth**.
- ❑ **Extended Convex Lifting (ECL)** reveals benign nonconvexity.



- ❑ **Global Optimality**

Local Stationarity

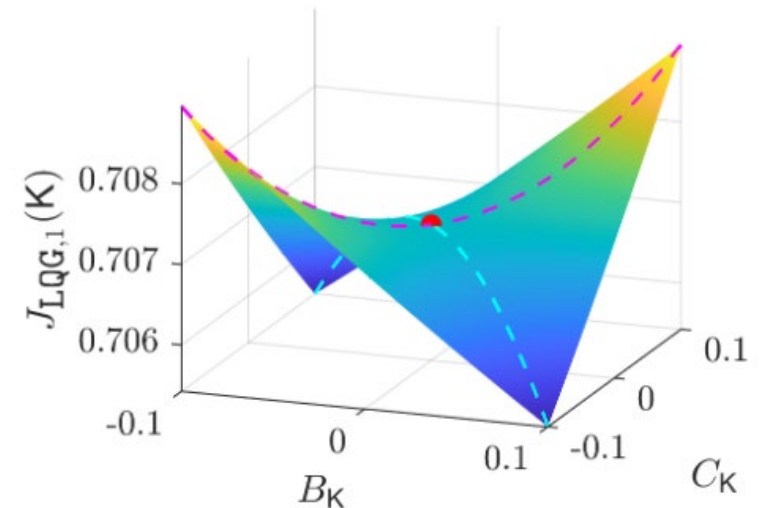
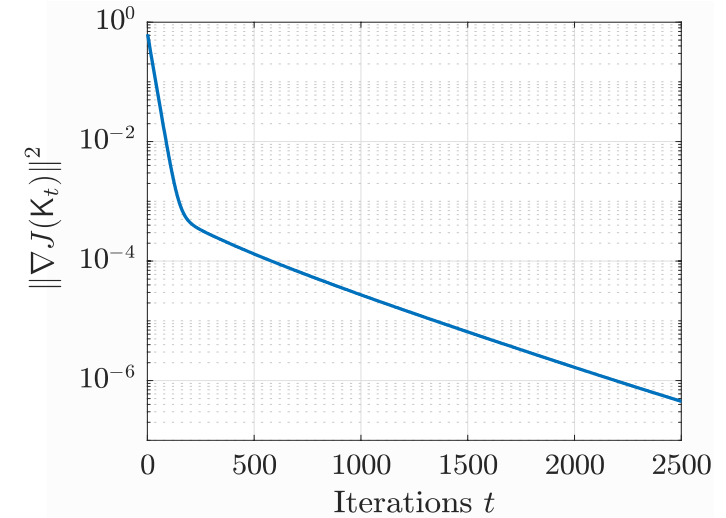
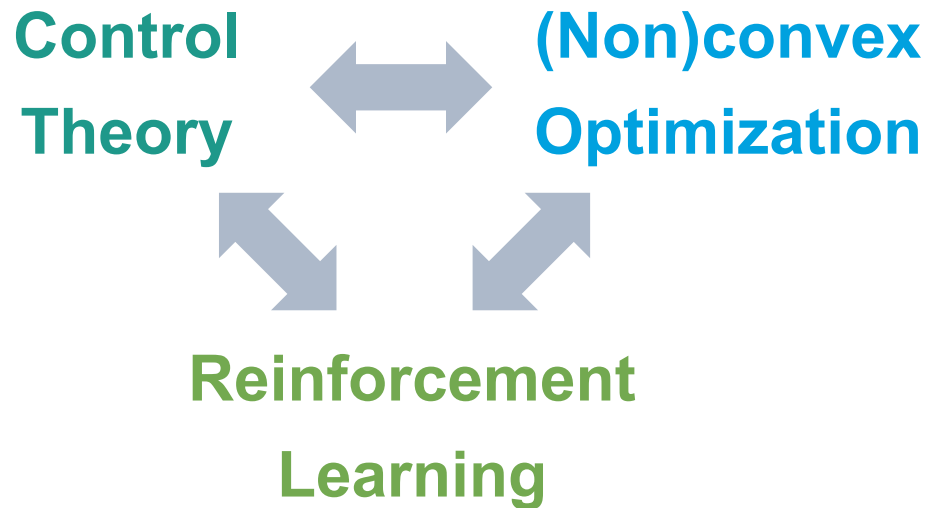


Structural
Information

Global Optimality
Certificate

Ongoing and Future work

- ❑ How to design efficient local search algorithms?
- ❑ How to establish convergence conditions and speeds?
- ❑ How to deal with degenerate points in local policy search? Avoiding saddle points with guarantees?
- ❑ How to establish guarantees beyond linear control?



Selected references

- Yujie Tang, Yang Zheng, and Na Li. "Analysis of the optimization landscape of Linear Quadratic Gaussian (LQG) control." *Mathematical Programming* 202.1 (2023): 399-444.
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- Yang Zheng, Chih-Fan Pai, and Yujie Tang. "Benign Nonconvex Landscapes in Optimal and Robust Control, Part II: Extended Convex Lifting." arXiv preprint arXiv:2406.04001 (2024): <https://arxiv.org/abs/2406.04001>
- Yuto Watanabe, and Yang Zheng. "Revisiting Strong Duality, Hidden Convexity, and Gradient Dominance in the Linear Quadratic Regulator." arXiv preprint arXiv:2503.10964 (2025): <https://arxiv.org/abs/2503.10964>
- Yuto Watanabe, Feng-Yi Liao, and Yang Zheng. "Policy Optimization in Robust Control: Weak Convexity and Subgradient Methods." *arXiv preprint arXiv:2509.25633* (2025). <https://arxiv.org/abs/2509.25633>

Thank you for your attention!

Q & A