Scalable Semidefinite and Polynomial Optimization via Matrix Decomposition

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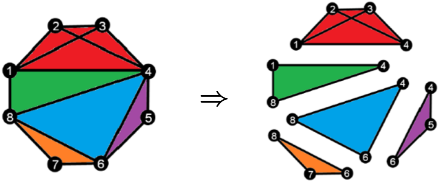
Outline

Introduction: Matrix decomposition and chordal graphs



Matrix decomposition and chordal graphs

Matrix decomposition:

- A simple example

$$A = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0}$$

- This is true for any PSD matrix with such pattern, *i.e.*, sparse cone decomposition

$$\underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}$$

where $*$ denotes a real scalar number (or block matrix).

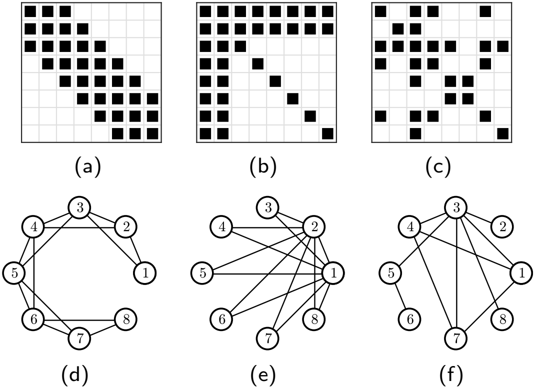
Benefits:

- Reduce computational complexity, and thus improve efficiency! $(3 \times 3 \to 2 \times 2)$

Matrix decomposition and chordal graphs

Matrix decomposition:

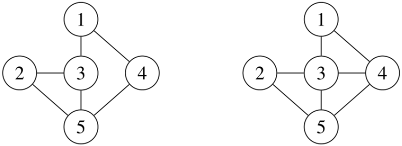
- Many other patterns admit similar decompositions, *e.g.*



- They can be commonly characterized by **chordal graphs**.

Chordal graphs

Chordal graphs: An undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *chordal* if every cycle of length greater than three has a chord.

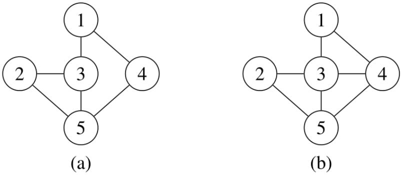


(a) (b)

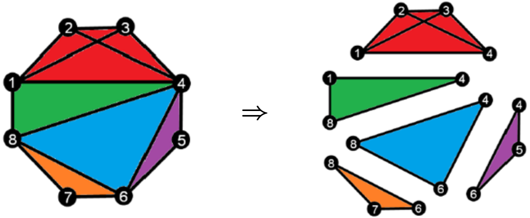
Notation: (Vandenberghe & Andersen, 2014)

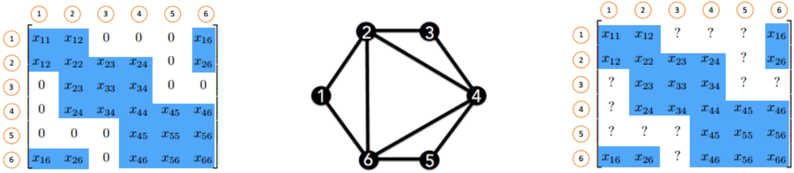
- *Maximal clique:* A clique is a set of nodes that induces a complete subgraph;
- *Clique decomposition:* A chordal graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ can be decomposed into a set of maximal cliques $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t\}$.

Chordal graphs: An undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *chordal* if every cycle of length greater than three has a chord.



Clique decomposition:





Sparse positive semidefinite (PSD) matrices

$$\mathbb{S}^n(\mathcal{E},0) = \{X \in \mathbb{S}^n \mid X_{ij} = X_{ji} = 0, \forall (i,j) \notin \mathcal{E}\},$$
$$\mathbb{S}^n_+(\mathcal{E},0) = \{X \in \mathbb{S}^n(\mathcal{E},0) \mid X \succeq 0\}.$$

Positive semidefinite completable matrices

$$\mathbb{S}^n(\mathcal{E},?) = \{X \in \mathbb{S}^n \mid X_{ij} = X_{ji}, \text{ given if } (i,j) \in \mathcal{E}\},$$
$$\mathbb{S}^n_+(\mathcal{E},?) = \{X \in \mathbb{S}^n(\mathcal{E},?) \mid \exists M \succeq 0, M_{ij} = X_{ij}, \forall (i,j) \in \mathcal{E}\}.$$

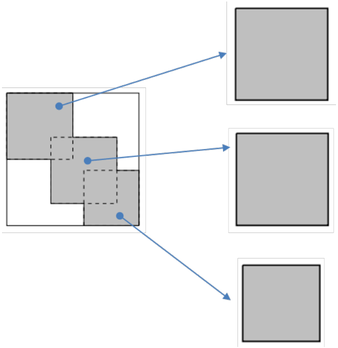
$\mathbb{S}^n_+(\mathcal{E},0)$ and $\mathbb{S}^n_+(\mathcal{E},?)$ are dual to each other.

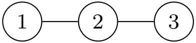
Two matrix decomposition theorems

Clique decomposition for PSD completable matrices (Grone, *et al.*, 1984)

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a chordal graph with maximal cliques $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. Then,

$$X \in \mathbb{S}^n_+(\mathcal{E}, ?) \Leftrightarrow E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^{\mathsf{T}} \in \mathbb{S}^{|\mathcal{C}_k|}_+, \qquad k = 1, \ldots, p.$$





$$\begin{bmatrix} X_{11} & X_{12} & ? \\ X_{21} & X_{22} & X_{23} \\ ? & X_{32} & X_{33} \end{bmatrix} \in \mathbb{S}^3_+(\mathcal{E}, ?)$$

$$\Updownarrow$$

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

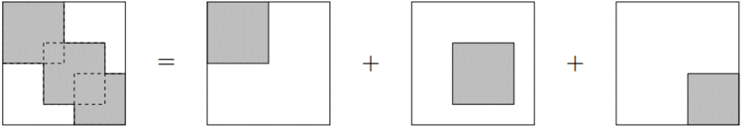
$$\begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix} \succeq 0$$

Two matrix decomposition theorems

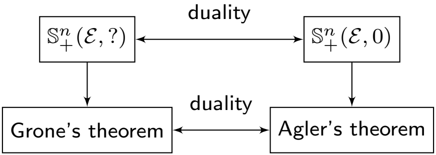
Clique decomposition for PSD matrices (Agler, Helton, McCullough, & Rodman, 1988; Griewank and Toint, 1984)

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a chordal graph with maximal cliques $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. Then,

$$Z \in \mathbb{S}^n_+(\mathcal{E}, 0) \Leftrightarrow Z = \sum_{k=1}^{p} E_{\mathcal{C}_k}^{\mathsf{T}} Z_k E_{\mathcal{C}_k}, \; Z_k \in \mathbb{S}_+^{|\mathcal{C}_k|}$$



Sparse Cone Decomposition



A growing number of applications

Control, machine learning, relaxation of QCQP, fluid dynamics, and beyond

Area	Topic	References
Control	Linear system analysis	Andersen et al. (2014b); Deroo et al. (2015); Mason & Papachristodoulou (2014); Pakazad et al. (2017b); Zheng et al. (2018c)
	Decentralized control	Deroo et al. (2014); Heinke et al. (2020); Zheng et al. (2020); Zheng et al. (2018d)
	Nonlinear system analysis	Schlosser & Korda (2020); Tacchi et al. (2019a); Zheng et al. (2019a); Mason (2015, Chapter 5)
	Model predictive control	Ahmadi et al. (2019); Hansson & Pakazad (2018)
Machine learning	Verification of neural networks	Batten et al. (2021); Dvijotham et al. (2020); Newton & Papachristodoulou (2021); Zhang (2020)
	Lipschitz constant estimation	Chen et al. (2020b); Latorre et al. (2020)
	Training of support vector machine	Andersen & Vandenberghe (2010)
	Geometric perception & coarsening	Chen et al. (2020a); Liu et al. (2019); Yang & Carlone (2020)
	Covariance selection	Dahl et al. (2008); Zhang et al. (2018)
	Subspace clustering	Miller et al. (2019a)
Relaxation of QCQP and POPs	Sensor network locations	Jing et al. (2019); Kim et al. (2009); Nie (2009)
	Max-Cut problem	Andersen et al. (2010a); Garstka et al. (2019); Zheng et al. (2020)
	Optimal power flow (OPF)	Andersen et al. (2014a); Dall'Anese et al. (2013); Jabr (2011); Jiang (2017); Molzahn & Hiskens (2014); Molzahn et al. (2013)
	State estimation in power systems	Weng et al. (2013); Zhang et al. (2017); Zhu & Giannakis (2014)
Others	Fluid dynamics	Arslan et al. (2021); Fantuzzi et al. (2018)
	Partial differential equations	Mevissen (2010); Mevissen et al. (2008, 2011, 2009)
	Robust quadratic optimization	Andersen et al. (2010b)
	Binary signal recovery	Fosson & Abuabiah (2019)
	Solving polynomial systems	Cifuentes & Parrilo (2016, 2017); Li et al. (2021); Mou et al. (2021); Tacchi et al. (2019b)
	Other problems	Baltean-Lugojan et al. (2019); Jeyakumar et al. (2016); Madani et al. (2017b); Pakazad et al. (2017a); Yang & Deng (2020)

This talk

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Chordal Graphs and Semidefinite Optimization

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Vision article

Chordal and factor-width decompositions for scalable semidefinite and polynomial optimization

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ABSTRACT

Chordal and factor-width decomposition methods for semidefinite programming and polynomial optimization have recently enabled the analysis and control of large-scale linear systems and medium-scale nonlinear systems. Chordal decomposition exploits the sparsity of semidefinite matrices in a semidefinite program (SDP), in order to formulate an equivalent SDP with smaller semidefinite constraints that can be solved more

- **Part I: Decomposition in sparse semidefinite optimization**
  - Zheng, Y., Fantuzzi, G., Papachristodoulou, A., Goulart, P., & Wynn, A. (2020). Chordal decomposition in operator-splitting methods for sparse semidefinite programs. Mathematical Programming, 180(1), 489-532.
- **Part II: Decomposition in sparse polynomial optimization**
  - Zheng, Y., Fantuzzi, G., & Papachristodoulou, A. (2019, July). Sparse sum-of-squares (SOS) optimization: A bridge between DSOS/SDSOS and SOS optimization for sparse polynomials. In 2019 American Control Conference (ACC) (pp. 5513-5518). IEEE..
- **Part III: Decomposition in polynomial matrix inequalities (PMIs)**
  - Zheng, Y., & Fantuzzi, G. (2021). Sum-of-squares chordal decomposition of polynomial matrix inequalities. Mathematical Programming, 1-38.

Part I: Decomposition in sparse semidefinite optimization

Semidefinite programs (SDPs)

$$\begin{aligned} \min \quad & \langle C, X\rangle \\ \text{subject to} \quad & \langle A_i, X\rangle = b_i, i = 1, \ldots, m, \\ & X \succeq 0. \end{aligned}$$

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y\rangle \\ \text{subject to} \quad & Z + \sum_{i=1}^{m} A_i\, y_i = C, \\ & Z \succeq 0. \end{aligned}$$

where $X \succeq 0$ means X is positive semidefinite.

- **Applications:** Control theory, fluid dynamics, polynomial optimization, *etc*.
- **Interior-point solvers:** SeDuMi, SDPA, SDPT3, MOSEK (suitable for small and medium-sized problems); *Modelling package:* YALMIP, CVX
- **Large-scale cases:** it is important to exploit inherent structures
  - Low rank;
  - Algebraic symmetry;
  - **Chordal sparsity**
    - Second-order methods: Fukuda *et al.*, 2001; Nakata *et al.*, 2003; Burer 2003; Andersen *et al.*, 2010.
    - **First-order methods**: Madani *et al.*, 2015; Sun, Andersen, and Vandenberghe, 2014.

Aggregate sparsity pattern of matrices

$$C = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, A_1 = \begin{bmatrix}2 & 1 & 0\\1 & 1 & 0\\0 & 0 & 2\end{bmatrix}, A_2 = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 2\end{bmatrix} \Longrightarrow \begin{bmatrix}* & * & 0\\* & * & *\\0 & * & *\end{bmatrix}$$

Primal SDP

$$\begin{aligned}\min \quad & \langle C, X\rangle\\ \text{subject to} \quad & \langle A_1, X\rangle = b_1\\ & \langle A_2, X\rangle = b_2\\ & X \succeq 0.\end{aligned}$$

Dual SDP

$$\begin{aligned}\max_{y,\, Z} \quad & \langle b, y\rangle\\ \text{subject to} \quad & y_1 A_1 + y_2 A_2 + Z = C,\\ & Z \succeq 0.\end{aligned}$$

$X \in \begin{bmatrix}* & * & ?\\* & * & *\\? & * & *\end{bmatrix}$	Patterns of feasible solutions	$Z \in \begin{bmatrix}* & * & 0\\* & * & *\\0 & * & *\end{bmatrix}$
$X \in \mathbb{S}^3_+(\mathcal{E}, ?)$	Cone replacement	$Z \in \mathbb{S}^3_+(\mathcal{E}, 0)$

Apply the clique decomposition on $\mathbb{S}^3_+(\mathcal{E}, ?)$ **and** $\mathbb{S}^3_+(\mathcal{E}, 0)$

- Fukuda *et al.*, 2001; Nakata *et al.*, 2003; Andersen *et al.*, 2010; Madani *et al.*, 2015; Sun, Andersen, and Vandenberghe, 2014.

Cone decomposition of sparse SDPs

Primal SDP

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{subject to} \quad & \langle A_i, X \rangle = b_i, i = 1, \ldots, m \\ & \boxed{X \succeq 0}. \end{aligned}$$

Dual SDP

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y \rangle \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i A_i + Z = C, \\ & \boxed{Z \succeq 0}. \end{aligned}$$

$X \in \mathbb{S}^n_+(\mathcal{E}, ?)$

Cone replacement

(Assuming an aggregate sparsity pattern $\mathcal{E}$)

$Z \in \mathbb{S}^n_+(\mathcal{E}, 0)$

$\Downarrow$

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, i = 1, \ldots, m \\ & \boxed{E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^{\mathsf{T}} \succeq 0, k = 1, \ldots, p}. \end{aligned}$$

$\Downarrow$

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y \rangle \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i A_i + \sum_{k=1}^{p} E_{\mathcal{C}_k}^{\mathsf{T}} Z_k E_{\mathcal{C}_k} = C, \\ & \boxed{Z_k \succeq 0, k = 1, \ldots, p} \end{aligned}$$

- **A big sparse PSD cone** is equivalently replaced by a set of **coupled small** PSD cones;
- Our idea: **consensus** variables $\Rightarrow$ decouple the coupling constraints;

Decomposed SDPs for operator-splitting algorithms

Primal decomposed SDP

$$\begin{aligned} \min_{X, X_k} \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \qquad i = 1, \ldots, m, \\ & \boxed{X_k = E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^{\mathsf{T}}, k = 1, \ldots, p,} \\ & X_k \in \mathbb{S}_+^{|\mathcal{C}_k|}, \qquad k = 1, \ldots, p. \end{aligned}$$

Dual decomposed SDP

$$\begin{aligned} \max_{y, Z_k, V_k} \quad & \langle b, y \rangle \\ \text{s.t.} \quad & \sum_{i=1}^{m} A_i \, y_i + \sum_{k=1}^{p} E_{\mathcal{C}_k}^{\mathsf{T}} V_k E_{\mathcal{C}_k} = C, \\ & \boxed{Z_k - V_k = 0, \ k = 1, \ldots, p,} \\ & Z_k \in \mathbb{S}_+^{|\mathcal{C}_k|}, \quad k = 1, \ldots, p. \end{aligned}$$

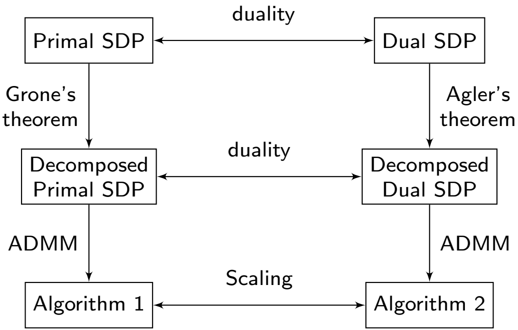
- A set of slack consensus variables has been introduced;
- The slack variables allow one to **separate the conic and the affine constraints** when using operator-splitting algorithms $\Rightarrow$ fast iterations:

projection on affine space
$+$ parallel projections on multiple small PSD cones
$\mathbb{S}_+^{|\mathcal{C}_k|}, k = 1, \ldots, p$

ADMM for primal and dual decomposed SDPs

Equivalence between the primal and dual cases

- ADMM steps in the dual form are scaled versions of those in the primal form.
- Extension to the homogeneous self-dual embedding exists.



Both algorithms only require conic projections onto small PSD cones. **Complexity depends on the largest maximal cliques, instead of the original dimension!**

Comparison with other first-order algorithms

Key difference: How to decouple the coupling constraints

Table 1: Comparison of first-order algorithms for solving SDPs. "Chordal Sparsity": whether the algorithm exploits chordal sparsity; "SDP Type": the types of SDP problems the algorithm considers; "Algorithm": the underlying first-order algorithm; "infeas./unbounded": whether the algorithm can detect infeasible or unbounded cases; "Solver": whether the code is open-source.

Reference	Chordal Sparsity	SDP Type	Algorithm	Infeas./ Unbounded	Solver
Wen et al. (2010)	✗	(3.2)	ADMM	✗	✗
Zhao et al. (2010)	✗	(3.2)	Augm. Lagrang.	✗	SDPNAL
O'Donoghue et al. (2016)	✗	(3.1)-(3.2)	ADMM	✓	SCS
Yurtsever et al. (2021)	✗	(3.1)[1]	SketchyCGAL	✗	CGAL
Lu et al. (2007)	✓	(3.1)	Mirror-Prox	✗	✗
Lam et al. (2012)	✓	OPF[2]	Primal-dual	✗	✗
Dall'Anese et al. (2013)	✓	OPF[2]	ADMM	✗	✗
Sun et al. (2014)	✓	Special[3]	Gradient proj.	✗	✗
Sun & Vandenberghe (2015)	✓	(3.1)-(3.2)	Spingarn	✗	✗
Kalbat & Lavaei (2015)	✓	Special[4]	ADMM	✗	✗
Madani et al. (2017a)	✓	General[5]	ADMM	✗	✗
Zheng et al. (2020)	✓	(3.1)-(3.2)	ADMM	✓	CDCS
Garstka et al. (2019)	✓	Quad. SDP[6]	ADMM	✓	COSMO

Note: 1. It requires an explicit trace constraint on X; 2. Special SDPs from the optimal power flow (OPF) problem; 3. Special SDPs from the matrix nearness problem; 4. Special SDPs with decoupled affine constraints; 5. General SDPs with inequality constraints; 6. A dual SDP (3.2) with a quadratic objective function.

CDCS

Cone decomposition conic solver

- An open source MATLAB solver for sparse conic programs (Julia interface);
- CDCS supports constraints on the following cones:
  - Free variables
  - non-negative orthant
  - second-order cone
  - the positive semidefinite cone.
- Input-output format: SeDuMi; Interface via YALMIP, SOSTOOLS.
- Syntax: `[x,y,z,info] = cdcs(At,b,c,K,opts);`

Download from `https://github.com/OxfordControl/CDCS`

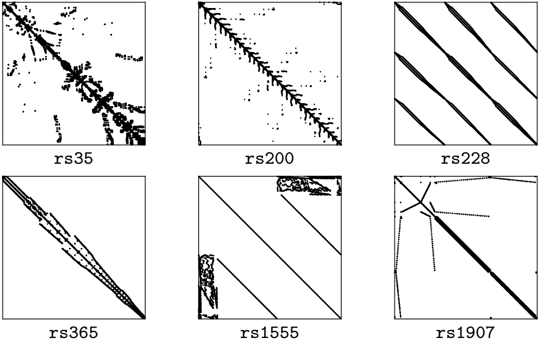
Numerical comparison

- SeDuMi (interior-point solver): default parameters, and low-accuracy solution 10^{-3}
- SCS (first-order solver)
- CDCS and SCS: stopping condition 10^{-3} (max. iterations 2000)
- All simulations were run on a PC with a 2.8 GHz Intel Core i7 CPU and 8GB of RAM.

Large-scale sparse SDPs

Instances from Andersen, Dahl, Vandenberghe, 2010

	rs35	rs200	rs228	rs365	rs1555	rs1907
Original cone size, n	2003	3025	1919	4704	7479	5357
Affine constraints, m	200	200	200	200	200	200
Number of cliques, p	588	1635	783	1244	6912	611
Maximum clique size	418	102	92	322	187	285
Minimum clique size	5	4	3	6	2	7



Large-scale sparse SDPs: Numerical results

	rs35			rs200		
	Time (s)	# Iter.	Objective	Time (s)	# Iter.	Objective
SeDuMi (high)	1 391	17	25.33	4 451	17	99.74
SeDuMi (low)	986	11	25.34	2 223	8	99.73
SCS (direct)	2 378	†2 000	25.08	9 697	†2 000	81.87
CDCS-primal	370	379	25.27	159	577	99.61
CDCS-dual	272	245	25.53	103	353	99.72
CDCS-hsde	208	198	25.64	54	214	99.77

	rs228			rs365		
	Time (s)	# Iter.	Objective	Time (s)	# Iter.	Objective
SeDuMi (high)	1 655	21	64.71	***	***	***
SeDuMi (low)	809	10	64.80	***	***	***
SCS (direct)	2 338	†2 000	62.06	34 497	†2 000	44.02
CDCS-primal	94	400	64.65	321	401	63.37
CDCS-dual	84	341	64.76	240	265	63.69
CDCS-hsde	38	165	65.02	151	175	63.75

	rs1555			rs1907		
	Time (s)	# Iter.	Objective	Time (s)	# Iter.	Objective
SeDuMi (high)	***	***	***	***	***	***
SeDuMi (low)	***	***	***	***	***	***
SCS (direct)	139 314	†2 000	34.20	50 047	†2 000	45.89
CDCS-primal	1 721	†2 000	61.22	330	349	62.87
CDCS-dual	317	317	69.54	271	252	63.30
CDCS-hsde	361	448	66.38	190	187	63.15

***: the problem could not be solved due to memory limitations.

†: maximum number of iterations reached.

Large-scale sparse SDPs: Numerical results

Average CPU time per iteration

	rs35	rs200	rs228	rs365	rs1555	rs1907
SCS (direct)	1.188	4.847	1.169	17.250	69.590	25.240
CDCS-primal	0.944	0.258	0.224	0.715	0.828	0.833
CDCS-dual	1.064	0.263	0.232	0.774	0.791	0.920
CDCS-hsde	1.005	0.222	0.212	0.733	0.665	0.891

- $20\times, 21\times, 26\times$, and $75\times$ faster than SCS, respectively, for problems `rs200`, `rs365`, `rs1907`, and `rs1555`.
- The computational benefit comes form the cone decomposition (projections onto small PSD cones)
- CDCS enables us to solve large, sparse conic problems with moderate accuracy that are beyond the reach of standard interior-point and/or other first-order methods

The conic projections in all Algorithms require $\mathcal{O}(\sum_{k=1}^{p} |\mathcal{C}_k|^3)$ flops. **Complexity is dominated by the largest maximal clique!**

Part II: Decomposition in sparse polynomial optimization

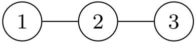
— bridging the gap between DSOS/SDSOS optimization and SOS optimization

Positive (semi)-definite polynomial matrices

- Recall the simple example

$$A = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0}$$

- **How about positive (semi)-definite polynomial matrices?**



$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & 0 \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ 0 & p_{32}(x) & p_{33}(x) \end{bmatrix} \succeq 0, \quad \forall x \in \mathcal{K}$$

$$\mathcal{K} = \mathbb{R}^n, \text{or}, \mathcal{K} = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

- **Point-wise:** the decomposition still holds, but can it be represented by polynomials or even better, by SOS matrices?

$$\underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}, \qquad \forall x \in \mathcal{K}$$

Checking nonnegativity and Sum-of-squares

Checking whether a given polynomial is nonnegative has applications in many areas.

$$p(x) = \sum p_\alpha x^\alpha \geq 0, \qquad \text{e.g.,} \quad p(x) = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_2^2 \geq 0.$$

- **Application**: unconstrained polynomial optimization

$$\min_{x \in \mathbb{R}^n} p(x) \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \geq 0. \end{aligned}$$

- **Sum-of-squares (SOS) relaxation**: $p(x)$ can be represented as a sum of finite squared polynomials $f_i(x), i = 1, \ldots, m$

$$p(x) = \sum_{i=1}^{m} f_i(x)^2,$$

- **SDP characterization (Parrilo 2000):** $p(x)$ is SOS if and only if

$$p(x) = v_d(x)^{\mathsf{T}} Q v_d(x), \qquad Q \succeq 0,$$

where $v_d(x) = [1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d]^{\mathsf{T}}$ is the standard monomial basis.

Checking nonnegativity and Sum-of-squares

Sum-of-square matrices

- Positive semidefinite polynomial matrices

$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \dots & p_{1r}(x) \\ p_{21}(x) & p_{22}(x) & \dots & p_{2r}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1}(x) & p_{r2}(x) & \dots & p_{rr}(x) \end{bmatrix} \succeq 0, \quad \forall x \in \mathbb{R}^n.$$

- The problem of checking whether $P(x) \succeq 0, \forall x \in \mathbb{R}^n$ is NP-hard.
- **SOS relaxation:** We call $P(x)$ is an SOS matrix if

$$p(x,y) = y^{\mathsf{T}} P(x) y \text{ is SOS in } [x; y]$$

- **SDP characterization** (Parrilo *et al.*): $P(x)$ is an SOS matrix if and only if

$$P(x) = (I_r \otimes v_d(x))^{\mathsf{T}} Q (I_r \otimes v_d(x)), \qquad Q \succeq 0.$$

SOS optimization

- **Scalar version:** given polynomials $p_0(x), p_h(x), h = 1, \ldots, t$, consider

$$\begin{aligned} \min_{u} \quad & w^{\mathsf{T}} u \\ \text{subject to} \quad & p_0(x) + \sum_{h=1}^{t} u_h p_h(x) \ \text{ is SOS.} \end{aligned}$$

- **Matrix version:** given polynomial matrices $P_0(x), P_h(x), h = 1, \ldots, t$, consider

$$\begin{aligned} \min_{u} \quad & w^{\mathsf{T}} u \\ \text{subject to} \quad & P_0(x) + \sum_{h=1}^{t} u_h P_h(x) \ \text{ is SOS.} \end{aligned}$$

- One fundamental problem is the poor scalability to large-scale instances, since

$$\binom{n+d}{d} = \mathcal{O}(n^d).$$

Scaled-diagonally dominant SOS (SDSOS) and DSOS

A new concept of (S)DSOS by Ahmadi and Majumdar, 2017

- *Diagonally dominant (dd) matrix:* a symmetric matrix $A = [a_{ij}]$ is dd if

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}|, \forall i = 1, \ldots, n.$$

- *Scaled-diagonally dominant (sdd) matrix:* a symmetric matrix $A = [a_{ij}]$ is sdd if there exists a PSD diagonal matrix D, such that

$$DAD \text{ is dd.}$$

- *DSOS polynomials:* $p(x) = v_d(x)^\mathsf{T} Q v_d(x)$, where Q is dd.
- *SDSOS polynomials:* $p(x) = v_d(x)^\mathsf{T} Q v_d(x)$, where Q is sdd.

LP and SOCP-based optimization (Ahmadi and Majumdar, 2017)

- Linear optimization over dd matrices or DSOS polynomials is a linear program (LP).
- Linear optimization over sdd matrices or SDSOS polynomials is a second-order cone program (SOCP).

The gap between DSOS/SDSOS and SOS

A brief summary

- **SOS:** $p(x) = v_d(x)^T Q v_d(x)$, where Q is PSD $\longrightarrow$ SDP
- **SDSOS:** $p(x) = v_d(x)^T Q v_d(x)$, where Q is sdd $\longrightarrow$ SOCP
- **DSOS:** $p(x) = v_d(x)^T Q v_d(x)$, where Q is dd $\longrightarrow$ LP

Another viewpoint

- **SDP** – PSD constraints of dimension $N \times N$
- **SOCP** – PSD constraints of dimension 2×2
- **LP** – PSD constraints of dimension 1×1

What is missing? How about problems that involve PSD constraints of dimension $k \times k$, where $1 \leq k \leq N$

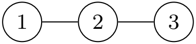
- Factor-width k matrices (Boman, et al. 2005) $\longrightarrow$ Not practical $\binom{n}{k} = \mathcal{O}(n^k)$
- **Chordal decomposition** $\longrightarrow$ the main topic today.

Sparse polynomial matrix (similar to sparse real matrix)

- Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, we define a sparse polynomial matrix $P(x)$ where

$$p_{ij}(x) = 0, \text{ if } (i,j) \notin \mathcal{E}^*$$

- For example, for a line graph of three nodes



$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ & p_{32}(x) & p_{33}(x) \end{bmatrix}.$$

- Define a set of sparse polynomial matrices

$$\mathbb{R}^r_{n,2d}(\mathcal{E}, 0) = \left\{ P(x) \in \mathbb{R}[x]^r_{n,2d} \middle| p_{ij}(x) = p_{ji}(x) = 0, \text{ if } (i,j) \notin \mathcal{E}^* \right\}.$$

- SOS/SDSOS/DSOS matrices with a sparsity pattern $\mathcal{E}$

$$\begin{aligned} SOS(\mathcal{E}, 0) &= SOS \cap \mathbb{R}^r_{n,2d}(\mathcal{E}, 0), \\ SDSOS(\mathcal{E}, 0) &= SDSOS \cap \mathbb{R}^r_{n,2d}(\mathcal{E}, 0), \\ DSOS(\mathcal{E}, 0) &= DSOS \cap \mathbb{R}^r_{n,2d}(\mathcal{E}, 0). \end{aligned}$$

Sparsity in SOS optimization

Sparsity in $P(x)$ does not necessarily lead to sparsity in Q.

$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ & p_{32}(x) & p_{33}(x) \end{bmatrix} = \begin{bmatrix} v(x)^\mathsf{T} Q_{11} v(x) & v(x)^\mathsf{T} Q_{12} v(x) & v(x)^\mathsf{T} Q_{13} v(x) \\ v(x)^\mathsf{T} Q_{21} v(x) & v(x)^\mathsf{T} Q_{22} v(x) & v(x)^\mathsf{T} Q_{23} v(x) \\ v(x)^\mathsf{T} Q_{31} v(x) & v(x)^\mathsf{T} Q_{32} v(x) & v(x)^\mathsf{T} Q_{33} v(x) \end{bmatrix}$$

$$= (I_3 \otimes v(x))^\mathsf{T} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} (I_3 \otimes v(x))$$

- We make **a restriction that** $Q_{ij} = 0$**, whenever** $p_{ij}(x) = 0$.
- Now, **chordal decomposition** on the sparse Q leads to

$$Q = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}$$

- We have the same chordal decomposition for polynomial matrix $P(x)$.

Sparse SOS matrix decomposition

Sparse version of SOS matrices

$$SSOS(\mathcal{E},0) = \Big\{ P(x) \in SOS(\mathcal{E},0) \,\Big|\, P(x) \text{ admits a } Q \succeq 0, \text{ with } Q_{ij} = 0 \text{ when } p_{ij}(x) = 0 \Big\}.$$

Theorem (Sparse SOS matrix decomposition)

If $\mathcal{E}$ is chordal with a set of maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_t$, then

$$P(x) \in SSOS(\mathcal{E},0) \Leftrightarrow P(x) = \sum_{k=1}^{\mathsf{T}} E_k^{\mathsf{T}} P_k(x) E_k,$$

where $P_k(x)$ is an SOS matrix of dimension $|\mathcal{C}_k| \times |\mathcal{C}_k|$.

LP/SOCP/SDP

We have the following inclusion relationship

$$DSOS(\mathcal{E},0) \subseteq SDSOS(\mathcal{E},0) \subseteq SSOS(\mathcal{E},0) \subseteq SOS(\mathcal{E},0) \subseteq \mathcal{P}(\mathcal{E},0)$$

- A brief summary (scalability):

$\mathcal{P}(\mathcal{E},0) \longrightarrow$ NP-hard

$DSOS(\mathcal{E},0) \longrightarrow$ LP (PSD cones: 1×1)

$SDSOS(\mathcal{E},0) \longrightarrow$ SOCP (PSD cones: 2×2)

$SSOS(\mathcal{E},0) \longrightarrow$ **SDP with smaller PSD cones of $k \times k$**

$SOS(\mathcal{E},0) \longrightarrow$ SDP with a PSD cone of $N \times N$

Solution quality: $\mathcal{P}_{\text{dsos}}, \mathcal{P}_{\text{sdsos}}$ and $\mathcal{P}_{\text{ssos}}$ are a sequence of inner approximations with increasing accuracy to the SOS problem $\mathcal{P}_{\text{sos}}$, meaning that

$$f^*_{\text{dsos}} \geq f^*_{\text{sdsos}} \geq f^*_{\text{ssos}} \geq f^*_{\text{sos}},$$

- Similar results can be shown for scalar sparse SOS optimization, which rely on the notion of *correlative sparsity pattern* (Waki *et al.*, 2006).

Implementations and numerical comparison

Packages

- SOS optimization: SOSTOOLS, YALMIP
- DSOS/SDSOS optimization: SPOTLESS
- Chordal decomposition: YALMIP (we adapted the option of correlative sparsity technique)
- SDP solver: Mosek

Numerical examples and applications

- Polynomial optimization problems
- Copositive optimization
- Control application: finding Lyapunov functions

Example 1: Polynomial optimization problems

Eigenvalue bounds on matrix polynomials

$$\min_{\gamma} \quad \gamma$$
$$\text{subject to} \quad P(x) + \gamma I \succeq 0,$$

where $n = 2, 2d = 2$, the polynomial is randomly generated. $P(x)$ has an arrow pattern.

Table: CPU time (in seconds) required by Mosek

Dimension r	10	20	30	40	50	60	70	80
SOS	0.30	1.33	6.64	27.3	108.1	308.7	541.3	1 018.6
SSOS	0.34	0.34	0.35	0.35	0.33	0.32	0.32	0.33
SDSOS	0.47	0.63	1.09	1.29	2.67	3.70	4.40	6.02
DSOS	**	**	**	**	**	**	**	**

**: The program is infeasible.

Example 1: Polynomial optimization problems

Eigenvalue bounds on matrix polynomials

$$\begin{array}{ll} \min_{\gamma} & \gamma \\ \text{subject to} & P(x) + \gamma I \succeq 0, \end{array}$$

where $n = 2, 2d = 2$, the polynomial is randomly generated. $P(x)$ has an arrow pattern.

Table: Optimal value γ

Dimension r	10	20	30	40	50	60	70	80
SOS	1.447	4.813	5.917	4.154	21.61	10.09	7.364	10.19
SSOS	1.454	4.878	5.917	4.498	21.64	12.71	7.558	11.39
SDSOS	40.1	279.3	1 254.4	145.5	762.8	1 521.1	1 217.3	598.0
DSOS	**	**	**	**	**	**	**	**

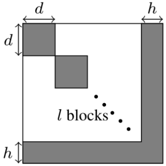
**: The program is infeasible.

Example 2: Copositive optimization

Consider the following copositive program

$$\begin{aligned} \min_{\gamma} \quad & \gamma \\ \text{subject to} \quad & Q + \gamma I \in \mathcal{C}^n, \end{aligned}$$

where Q is a random symmetric matrix with a block-arrow sparsity pattern.



Numerical results

The block size is $d = 3$; arrow head is $h = 2$; we vary the number of blocks l

Table: CPU time (in seconds) required by Mosek

l	2	4	6	8	10
SOS	0.45	7.34	248.9	*	*
SSOS	0.39	0.41	0.38	0.49	0.40
SDSOS	0.54	1.22	4.99	11.07	32.18
DSOS	0.59	0.76	2.19	5.72	17.11

*: Out of memory.

Example 2: Copositive optimization

Consider the following copositive program

$$\begin{aligned} \min_{\gamma} \quad & \gamma \\ \text{subject to} \quad & Q + \gamma I \in \mathcal{C}^n, \end{aligned}$$

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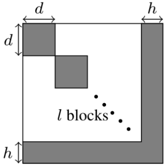


Numerical results

The block size is $d = 3$; arrow head is $h = 2$; we vary the number of blocks l

Table: Optimal value γ

l	2	4	6	8	10
SOS	1.137	4.197	2.836	*	*
SSOS	1.137	4.197	2.836	4.043	4.718
SDSOS	1.184	4.500	3.282	4.562	5.146
DSOS	2.551	7.775	6.452	12.057	15.203

*: Out of memory.

Example 3: Finding Lyapunov functions

Control application: finding Lyapunov functions

- Consider a dynamical system with a banded pattern

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2), & g_1(x) &= \gamma - x_1^2 \geq 0 \\
\dot{x}_2 &= f_2(x_1, x_2, x_3), & g_2(x) &= \gamma - x_2^2 \geq 0 \\
&\vdots \\
\dot{x}_n &= f_n(x_{n-1}, x_n), & g_2(x) &= \gamma - x_n^2 \geq 0
\end{aligned}
$$

- Generate locally stable systems of degree three;
- Consider a polynomial Lyapunov function of degree two with a banded pattern

$$V(x) = V_1(x_1, x_2) + V_2(x_1, x_2, x_3) + \ldots + V_n(x_{n-1}, x_n)$$

- Then, we consider the following SOS program

$$
\begin{aligned}
\text{Find} \quad & V(x), r_i(x) \\
\text{subject to} \quad & V(x) - \epsilon(x^{\mathsf{T}} x) \text{ is SOS} \\
& -\langle \nabla V(x), f(x) \rangle - \sum_{i=1}^{n} r_i(x) g_i(x) \text{ is SOS} \\
& r_i(x) \text{ is SOS}, i = 1, \ldots, n.
\end{aligned}
$$

Example 3: Finding Lyapunov functions

Control application: finding Lyapunov functions

Table: CPU time (in seconds) required by Mosek

n	10	15	20	30	40	50
SOS	1.29	18.44	247.84	*	*	*
SSOS	0.55	0.68	0.71	0.83	1.04	1.17
SDSOS	0.71	1.76	4.47	32.21	85.99	257.20
DSOS	0.70	1.42	3.58	35.12	73.64	324.32

*: Out of memory.

Part III: Decomposition in positive semidefinite polynomial matrices

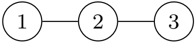
— sparsity-exploiting versions of the Hilbert-Artin, Reznick, Putinar, and Putinar-Vasilescu Positivstellensätze.

Positive (semi)-definite polynomial matrices

- Recall the simple example

$$A = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0}$$

- **How about positive (semi)-definite polynomial matrices?**



$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & 0 \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ 0 & p_{32}(x) & p_{33}(x) \end{bmatrix} \succeq 0, \quad \forall x \in \mathcal{K}$$

$$\mathcal{K} = \mathbb{R}^n, \text{or}, \mathcal{K} = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$$

- **Point-wise:** the decomposition still holds, but can it be represented by polynomials or even better, by SOS matrices?

$$\underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}, \qquad \forall x \in \mathcal{K}$$

Naive extension does not work

Negative result

There exists an n-variate r polynomial matrix $P(x)$ with chordal sparsity $\mathcal{G}$ that is strictly positive definite for all $x \in \mathbb{R}^n$, but cannot be written as the decomposition form with positive semidefinite polynomial matrices $S_k(x)$.

- **Example:**

$$P(x) = \begin{bmatrix} k+1+x^2 & x+x^2 & 0 \\ x+x^2 & k+2x^2 & x-x^2 \\ 0 & x-x^2 & k+1+x^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ x & x \\ 1 & -x \end{bmatrix} \begin{bmatrix} x & x & 1 \\ 1 & x & -x \end{bmatrix} + kI_3$$

- It is not difficult to show that

$$P(x) = \underbrace{\begin{bmatrix} a(x) & b(x) & 0 \\ b(x) & c(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & d(x) & e(x) \\ 0 & e(x) & f(x) \end{bmatrix}}_{\succeq 0},$$

fails to exist when $0 \le k < 2$.

- $P(x)$ is strictly positive definite if $0 < k < 2$.

Hilbert–Artin theorem

Sparse matrix version of the Hilbert–Artin theorem

Let $P(x)$ be an $m \times m$ PSD polynomial matrix whose sparsity graph is chordal and has maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_t$. There exist an SOS polynomial $\sigma(x)$ and SOS matrices $S_k(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$\sigma(x)P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

- **Example:** $\sigma(x) = 1 + k + x^2$ suffices for the previous example

$$P(x) = \begin{bmatrix} k+1+x^2 & x+x^2 & 0 \\ x+x^2 & \dfrac{(1+x)^2x^2}{1+k+x^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{k^2+k+3kx^2+(1-x)^2x^2}{1+k+x^2} & x-x^2 \\ 0 & x-x^2 & k+1+x^2 \end{bmatrix}$$

- PSD polynomial matrices are equivalent to SOS matrices when $n = 1$.

Putinar's Positivstellensatz

Consider $P(x) \succ 0, \forall x \in \mathcal{K}$ with $\mathcal{K} = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m\}$, and

$$\sigma_0(x) + g_1(x)\sigma_1(x) + \cdots + g_q(x)\sigma_q(x) = r^2 - \|x\|^2.$$

Sparse matrix version of Putinar's Positivstellensatz

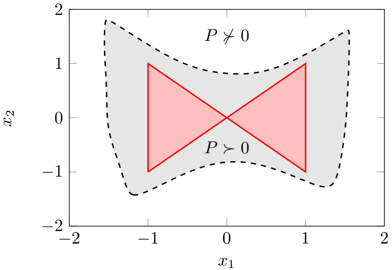
let $P(x)$ be a polynomial matrix whose sparsity graph is chordal and has maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_t$. If P is strictly positive definite on $\mathcal{K}$ (satisfying the Archimedean condition), there exist SOS matrices $S_{j,k}(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} \left(S_{0,k}(x) + \sum_{j=1}^{q} g_j(x) S_{j,k}(x) \right) E_{\mathcal{C}_k}.$$

- **Example:** Consider $\mathcal{K} = \{x \in \mathbb{R}^2 : \; g_1(x) := 1 - x_1^2 \geq 0, \; g_2(x) := x_1^2 - x_2^2 \geq 0\}$, and

$$P(x) := \begin{bmatrix} 1 + 2x_1^2 - x_1^4 & x_1 + x_1x_2 - x_1^3 & 0 \\ x_1 + x_1x_2 - x_1^3 & 3 + 4x_1^2 - 3x_2^2 & 2x_1^2x_2 - x_1x_2 - 2x_2^3 \\ 0 & 2x_1^2x_2 - x_1x_2 - 2x_2^3 & 1 + x_2^2 + x_1^2x_2^2 - x_2^4 \end{bmatrix}$$

Putinar's Positivstellensatz



- It guarantees the following decomposition holds for some SOS matrices $S_{i,j}(x)$

$$P(x) = \sum_{k=1}^{2} E_{\mathcal{C}_k}^{\mathsf{T}} \left[S_{0,k}(x) + g_1(x)S_{1,k}(x) + g_2(x)S_{2,k}(x)\right] E_{\mathcal{C}_k}$$

- Possible choices are

$$S_{0,1}(x) = I_2 + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \qquad S_{1,1}(x) = \begin{bmatrix} x_1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \end{bmatrix}$$

$$S_{0,2}(x) = I_2 + \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \begin{bmatrix} x_1 & -x_2 \end{bmatrix} \qquad S_{2,2}(x) = \begin{bmatrix} 2 \\ x_2 \end{bmatrix} \begin{bmatrix} 2 & x_2 \end{bmatrix}.$$

Application to robust semidefinite optimization

Consider a robust SDP program

$$B^* := \inf_{\lambda \in \mathbb{R}^\ell} \quad b^\mathsf{T} \lambda$$

$$\text{subject to} \quad P(x, \lambda) := P_0(x) - \sum_{i=1}^{\ell} P_i(x) \lambda_i \succeq 0 \quad \forall x \in \mathcal{K},$$

$$B^*_{d,\nu} := \inf_{\lambda,\, S_{j,k}} \quad b^\mathsf{T} \lambda$$

$$\text{subject to} \quad \sigma(x)^\nu P(x, \lambda) = \sum_{k=1}^{t} E^\mathsf{T}_{\mathcal{C}_k} \left(S_{0,k}(x) + \sum_{j=1}^{m} g_j(x) S_{j,k}(x) \right) E_{\mathcal{C}_k},$$

$$S_{j,k} \in \Sigma^{|\mathcal{C}_k|}_{2d_j} \quad \forall j = 0, \ldots, q,\ \forall k = 1, \ldots, t,$$

Convergence guarantees

- $\mathcal{K}$ is compact and satisfies the Archimedean condition, under some technical conditions, we fix $\sigma(x) = 1$ and $B^*_{d,0} \to B^*$ from above as $d \to \infty$.
- $\mathcal{K} \equiv \mathbb{R}^n$: under some technical conditions, we fix $\sigma(x) = 1 + \|x\|^2$ and $B^*_{d,\nu} \to B^*$ from above as $\nu \to \infty$ and $d = \nu + \lceil \frac{1}{2} \max\{\deg(P), \deg(g_1), \ldots, \deg(g_q)\} \rceil$.

Define a set

$$\mathcal{F}_\omega = \{\lambda \in \mathbb{R}^2 : \; P_\omega(x,\lambda) \succeq 0 \quad \forall x \in \mathbb{R}^3\}.$$

$$P_\omega(x,\lambda) = \begin{bmatrix} \lambda_2 x_1^4 + x_2^4 & \lambda_1 x_1^2 x_2^2 & & & & & & \\ \lambda_1 x_1^2 x_2^2 & \lambda_2 x_2^4 + x_3^4 & \lambda_2 x_2^2 x_3^2 & & & & & \\ & \lambda_2 x_2^2 x_3^2 & \lambda_2 x_3^4 + x_1^4 & \lambda_1 x_1^2 x_3^2 & & & & \\ & & \lambda_1 x_1^2 x_3^2 & \lambda_2 x_1^4 + x_2^4 & \lambda_2 x_1^2 x_2^2 & & & \\ & & & \lambda_2 x_1^2 x_2^2 & \lambda_2 x_2^4 + x_3^4 & \ddots & & \\ & & & & \ddots & \ddots & \lambda_i x_2^2 x_3^2 \\ & & & & & \lambda_i x_2^2 x_3^2 & \lambda_2 x_3^4 + x_1^4 \end{bmatrix},$$

- Define two hierarchies of subsets of $\mathcal{F}_\omega$, indexed by a nonnegative integer ν, as

$$\mathcal{D}_{\omega,\nu} := \left\{\lambda \in \mathbb{R}^2 : \; \|x\|^{2\nu} P_\omega(x,\lambda) \text{ is SOS}\right\},$$

$$\mathcal{S}_{\omega,\nu} := \left\{\lambda \in \mathbb{R}^2 : \; \|x\|^{2\nu} P_\omega(x,\lambda) = \sum_{k=1}^{3\omega-1} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}, S_k(x) \text{ is SOS}\right\}.$$

- We always have

$$\mathcal{S}_{\omega,\nu} \subseteq \mathcal{D}_{\omega,\nu} \subseteq \mathcal{F}_\omega$$

Experiment 1: global PMI

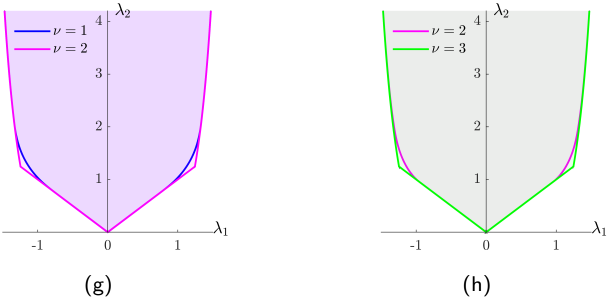


Figure: Inner approximations of the set $\mathcal{F}_2$ obtained with SOS optimization. (a) Sets $\mathcal{D}_{2,\nu}$ obtained using the standard SOS constraint; (b) Sets $\mathcal{S}_{2,\nu}$ obtained using the sparse SOS constraint. The numerical results suggest $\mathcal{S}_{2,3} = \mathcal{D}_{2,2} = \mathcal{F}_2$.

Numerical Experiments

We consider

$$B^* := \inf_{\lambda} \quad \lambda_2 - 10\lambda_1$$
$$\text{subject to} \quad \lambda \in \mathcal{F}_\omega$$

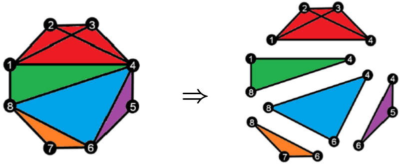
Table: Upper bounds $B_{d,\nu}$ on the optimal value B^* and CPU time (seconds) by MOSEK

	Standard SOS						Sparse SOS					
	$\nu = 1$		$\nu = 2$		$\nu = 3$		$\nu = 2$		$\nu = 3$		$\nu = 4$	
ω	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$	t	$B_{d,\nu}$
5	12	-8.68	25	-9.36	69	-9.36	0.58	-8.97	0.72	-9.36	1.29	-9.36
10	407	-8.33	886	-9.09	2910	-9.09	1.65	-8.72	0.82	-9.09	2.08	-9.09
15	2090	-8.26	OOM	OOM	OOM	OOM	2.76	-8.68	1.13	-9.04	2.79	-9.04
20	OOM	OOM	OOM	OOM	OOM	OOM	3.24	-8.66	1.54	-9.02	4.70	-9.02
25	OOM	OOM	OOM	OOM	OOM	OOM	2.85	-8.66	1.94	-9.02	4.59	-9.02
30	OOM	OOM	OOM	OOM	OOM	OOM	2.38	-8.65	2.40	-9.01	5.50	-9.01
35	OOM	OOM	OOM	OOM	OOM	OOM	2.66	-8.65	3.25	-9.01	6.17	-9.01
40	OOM	OOM	OOM	OOM	OOM	OOM	3.07	-8.65	3.14	-9.01	8.48	-9.01

Conclusion

Take-home message

- **Message 1: Chordal decomposition:** leading to sparse PSD cone decompositions



- **Message 2: Sparse SDPs can be solved 'fast'**

$$\begin{aligned} \min_{x, x_k} \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax = b, \\ & \boxed{x_k = H_k x}, \quad k = 1, \ldots, p, \\ & x_k \in \mathcal{S}_k, \quad k = 1, \ldots, p, \end{aligned}$$

$$\sigma(x)P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

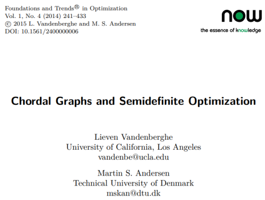
CDCS: an open-source first-order conic solver;

Download from `https://github.com/OxfordControl/CDCS`

- **Message 3: Sparse robust SDPs can be solved 'fast':** the Hilbert-Artin, Reznick, Putinar, and Putinar-Vasilescu Positivstellensätze.

Future work

- Decomposition and completion of polynomial matrices
- Moment interpretation of the PSD polynomial decomposition results
- Combining matrix decomposition with other structures
- Blending application-driven modeling with optimization
- Efficient software for modern computers



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Chordal Graphs and Semidefinite Optimization

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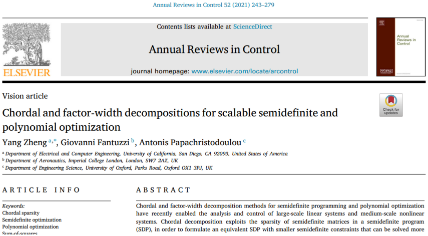

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Vision article

Chordal and factor-width decompositions for scalable semidefinite and polynomial optimization

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ABSTRACT

Chordal and factor-width decomposition methods for semidefinite programming and polynomial optimization have recently enabled the analysis and control of large-scale linear systems and medium-scale nonlinear systems. Chordal decomposition exploits the sparsity of semidefinite matrices in a semidefinite program (SDP), in order to formulate an equivalent SDP with smaller semidefinite constraints that can be solved more

Thank you for your attention!

Q & A

- Zheng, Y., Fantuzzi, G., Papachristodoulou, A., Goulart, P., & Wynn, A. (2020). Chordal decomposition in operator-splitting methods for sparse semidefinite programs. Mathematical Programming, 1-44.
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Extra slides

Alternating Direction Method of Multipliers (ADMM)

The ADMM algorithm solves the optimization problem (Bertsekas and Tsitsiklis, 1989; Boyd, *et al.*, 2011)

$$\begin{aligned} \min_{x,y} \quad & f(x) + g(y) \\ \text{subject to} \quad & Ax + By = c, \end{aligned}$$

where f and g are convex functions.

- **Augmented Lagrangian**

$$\mathcal{L}_\rho(x, y, z) := f(x) + g(y) + z^\mathsf{T}(Ax + By - c) + \frac{\rho}{2}\|Ax + By - c\|^2$$

- **ADMM steps**

$$\begin{aligned} x^{(n+1)} &= \arg\min_x \ \mathcal{L}_\rho(x, y^{(n)}, z^{(n)}), && \to x\text{-minimization step} \\ y^{(n+1)} &= \arg\min_y \ \mathcal{L}_\rho(x^{(n+1)}, y, z^{(n)}), && \to y\text{-minimization step} \\ z^{(n+1)} &= z^{(n)} + \rho\,(Ax^{(n+1)} + By^{(n+1)} - c). && \to \text{dual variable update} \end{aligned}$$

ADMM is particularly suitable when the subproblems have closed-form expressions, or can be solved efficiently.

ADMM for primal decomposed SDPs

$$\begin{aligned}\min_{x,x_k} \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax = b, \\ & \boxed{x_k = H_k x}, \quad k = 1, \ldots, p, \\ & x_k \in \mathcal{S}_k, \qquad k = 1, \ldots, p,\end{aligned}$$

Reformulation using indicator functions

$$\begin{aligned}\min_{x,x_1,\ldots,x_p} \quad & \langle c, x\rangle + \delta_0(Ax - b) + \sum_{k=1}^{p} \delta_{\mathcal{S}_k}(x_k) \\ \text{s.t.} \quad & x_k = H_k x, \quad k = 1, \ldots, p.\end{aligned}$$

- *x-minimization step:* QP with linear constraints, KKT condition

$$\begin{bmatrix} D & A^{\mathsf{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} H_k^{\mathsf{T}} \left(x_k^{(n)} + \rho^{-1}\lambda_k^{(n)} \right) - \rho^{-1} c \\ b \end{bmatrix}.$$

- *y-minimization step:* Parallel projections onto **small PSD cones**

$$\begin{aligned}\min_{x_k} \quad & \left\| x_k - H_k x^{(n+1)} + \rho^{-1}\lambda_k^{(n)} \right\|^2 \\ \text{s.t.} \quad & x_k \in \mathcal{S}_k.\end{aligned}$$

- Update multipliers

$$\begin{aligned}\max_{y, z_k, v_k} \quad & \langle b, y\rangle \\ \text{s.t.} \quad & A^{\mathsf{T}} y + \sum_{k=1}^{p} H_k^{\mathsf{T}} v_k = c, \\ & \boxed{z_k - v_k = 0}, k = 1, \ldots, p, \\ & z_k \in \mathcal{S}_k, k = 1, \ldots, p.\end{aligned}$$

Reformulation using indicator functions

$$\begin{aligned}\min \quad & -\langle b, y\rangle + \delta_0\left(c - A^{\mathsf{T}} y - \sum_{k=1}^{p} H_k^{\mathsf{T}} v_k\right) + \sum_{k=1}^{p} \delta_{\mathcal{S}_k}(z_k) \\ \text{s.t.} \quad & z_k = v_k, \quad k = 1, \ldots, p.\end{aligned}$$

- *x-minimization step:* QP with linear constraints, KKT condition

$$\begin{bmatrix} D & A^{\mathsf{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c - \sum_{k=1}^{p} H_k^{\mathsf{T}}\left(z_k^{(n)} + \rho^{-1}\lambda_k^{(n)}\right) \\ -\rho^{-1} b \end{bmatrix},$$

- *y-minimization step:* Parallel projections onto **small PSD cones**

$$\begin{aligned}\min_{z_k} \quad & \left\| z_k - v_k^{(n)} + \rho^{-1}\lambda_k^{(n)} \right\|^2 \\ \text{s.t.} \quad & z_k \in \mathcal{S}_k.\end{aligned}$$

- Update multipliers

Homogeneous self-dual embedding of decomposed SDPs

$$
\begin{aligned}
\min_{x, x_k} \quad & \langle c, x \rangle \\
\text{s.t.} \quad & Ax = b, \\
& x_k = H_k x, \quad k = 1, \ldots, p, \\
& x_k \in \mathcal{S}_k, \quad k = 1, \ldots, p,
\end{aligned}
$$

$$
\begin{aligned}
\max_{y, z_k, v_k} \quad & \langle b, y \rangle \\
\text{s.t.} \quad & A^{\mathsf{T}} y + \sum_{k=1}^{p} H_k^{\mathsf{T}} v_k = c, \\
& z_k - v_k = 0, k = 1, \ldots, p, \\
& z_k \in \mathcal{S}_k, k = 1, \ldots, p.
\end{aligned}
$$

Notional simplicity:

$$
s := \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \quad z := \begin{bmatrix} z_1 \\ \vdots \\ z_p \end{bmatrix}, \quad t := \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix}, \quad H := \begin{bmatrix} H_1 \\ \vdots \\ H_p \end{bmatrix}, \quad \mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_p
$$

KKT conditions

- Primal feasibility

$$
Ax^* - r^* = b, \quad s^* + w^* = Hx^*, \quad s^* \in \mathcal{S}, \quad r^* = 0, \qquad w^* = 0.
$$

- Dual feasibility

$$
A^{\mathsf{T}} y^* + H^{\mathsf{T}} t^* + h^* = c, \qquad z^* - t^* = 0, \qquad z^* \in \mathcal{S}, \qquad h^* = 0.
$$

- Zero duality gap:

$$
c^{\mathsf{T}} x^* - b^{\mathsf{T}} y^* = 0.
$$

Homogeneous self-dual embedding of decomposed SDPs

The homogeneous self-dual embedding (HSDE) form (Ye, Todd, Mizuno, 1994)

$$\begin{aligned} \text{find} \quad & (u, v) \\ \text{subject to} \quad & v = Qu, \\ & (u, v) \in \mathcal{K} \times \mathcal{K}^*, \end{aligned}$$

where $\mathcal{K} := \mathbb{R}^{n^2} \times \mathcal{S} \times \mathbb{R}^m \times \mathbb{R}^{n_d} \times \mathbb{R}_+$ is a cone ($\mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_p$) and

$$u := \begin{bmatrix} x \\ s \\ y \\ t \\ \tau \end{bmatrix}, \quad v := \begin{bmatrix} h \\ z \\ r \\ w \\ \kappa \end{bmatrix}, \quad Q := \begin{bmatrix} 0 & 0 & -A^\mathsf{T} & -H^\mathsf{T} & c \\ 0 & 0 & 0 & I & 0 \\ A & 0 & 0 & 0 & -b \\ H & -I & 0 & 0 & 0 \\ -c^\mathsf{T} & 0 & b^\mathsf{T} & 0 & 0 \end{bmatrix}.$$

ADMM steps (similar to the solver SCS, O'Donoghue *et al.*, 2016)

$$\hat{u}^{(n+1)} = (I + Q)^{-1} \left(u^{(n)} + v^{(n)} \right), \quad \longrightarrow \text{Projection onto a linear subspace}$$

$$u^{(n+1)} = \mathbb{P}_{\mathcal{K}} \left(\hat{u}^{(n+1)} - v^{(n)} \right), \quad \longrightarrow \text{Projection onto } \textbf{small PSD cones}$$

$$v^{(n+1)} = v^{(n)} - \hat{u}^{(n+1)} + u^{(n+1)}, \quad \longrightarrow \text{Computationally trivial update}$$

The conic projections in all Algorithms require $\mathcal{O}(\sum_{k=1}^{p} |\mathcal{C}_k|^3)$ flops.

Reznick's Positivstellensatz

Sparse matrix version of Reznick's Positivstellensatz

Let $P(x)$ be an $m \times m$ homogeneous polynomial matrix whose sparsity graph is chordal and has maximal cliques $\mathcal{C}_1, \dots, \mathcal{C}_t$. If P is strictly positive definite on $\mathbb{R}^n \setminus \{0\}$, there exist an integer $\nu \geq 0$ and homogeneous SOS matrices $S_k(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$\|x\|^{2\nu} P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

- **Corollary**: If P is strictly positive definite on $\mathbb{R}^n$ and its highest-degree homogeneous part $\sum_{|\alpha|=2d} P_\alpha x^\alpha$ is strictly positive definite on $\mathbb{R}^n \setminus \{0\}$, there exist an integer $\nu \geq 0$ and SOS matrices $S_k(x)$ of size $|\mathcal{C}_k| \times |\mathcal{C}_k|$ such that

$$(1 + \|x\|^2)^{\nu} P(x) = \sum_{k=1}^{t} E_{\mathcal{C}_k}^{\mathsf{T}} S_k(x) E_{\mathcal{C}_k}.$$

- **Non-trivial example:** Let $q(x) = x_1^2 x_2^4 + x_1^4 x_2^2 - 3x_1^2 x_2^2 + 1$ be the Motzkin polynomial, and

$$P(x) = \begin{bmatrix} 0.01(1 + x_1^6 + x_2^6) + q(x) & -0.01x_1 & 0 \\ -0.01x_1 & x_1^6 + x_2^6 + 1 & -x_2 \\ 0 & -x_2 & x_1^6 + x_2^6 + 1 \end{bmatrix}.$$

- $P(x)$ is is strictly positive definite on $\mathbb{R}^2$, but is not SOS (since $\varepsilon(1 + x_1^6 + x_2^6) + q(x)$ is not SOS unless $\varepsilon \gtrsim 0.01006$ [Laurent 2009, Example 6.25]).
- Our theorem guarantees the following decomposition exists

$$(1 + \|x\|^2)^\nu P(x) = E_{\mathcal{C}_1}^\intercal S_1(x) E_{\mathcal{C}_1} + E_{\mathcal{C}_2}^\intercal S_2(x) E_{\mathcal{C}_2}.$$

- It suffices to use $\nu = 1$ and SOS matrices

$$S_1(x) = \begin{bmatrix} (1 + \|x\|^2) q(x) & 0 \\ 0 & 0 \end{bmatrix} + \frac{1 + \|x\|^2}{100} \begin{bmatrix} 1 + x_1^6 + x_2^6 & -x_1 \\ -x_1 & 100x_1^2 \end{bmatrix},$$

$$S_2(x) = (1 + \|x\|^2) \begin{bmatrix} 1 - x_1^2 + x_1^6 + x_2^6 & -x_2 \\ -x_2 & 1 + x_1^6 + x_2^6 \end{bmatrix}.$$

Proof ideas: Hilbert–Artin theorem

Diagonalization with no fill-ins

If $P(x)$ is an $m \times m$ symmetric polynomial matrix with chordal sparsity graph, there exist an $m \times m$ permutation matrix T, an invertible $m \times m$ lower-triangular polynomial matrix $L(x)$, and polynomials $b(x)$, $d_1(x)$, $\ldots$, $d_m(x)$ such that

$$b^4(x)\, TP(x)T^{\mathsf{T}} = L(x)\mathrm{Diag}\left(d_1(x), \ldots, d_m(x)\right) L(x)^{\mathsf{T}}.$$

Moreover, L has no fill-in in the sense that $L + L^{\mathsf{T}}$ has the same sparsity as TPT^{T}.

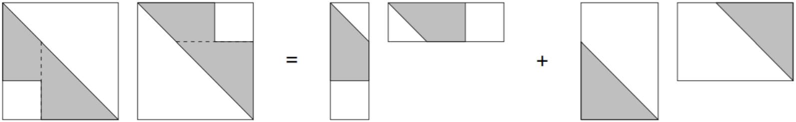


Figure: Decomposition follows by combining columns.

Figure from Prof. Lieven Vandenberghe's talk.

Proof ideas: Putinar's theorem

Scherer and Ho, 2006

Let $\mathcal{K}$ be a compact semialgebraic set that satisfies the Archimedean condition. If an $m \times m$ symmetric polynomial matrix $P(x)$ is strictly positive definite on $\mathcal{K}$, there exist $m \times m$ SOS matrices $S_0, \ldots, S_q$ such that

$$P(x) = S_0(x) + \sum_{i=1}^{q} S_i(x) g_i(x).$$

- Weierstrass polynomial approximation theorem + the above version of Putinar's Positivstellensatz

$$P(x) = \begin{bmatrix} a(x) & b(x)^{\mathsf{T}} & 0 \\ b(x) & U(x) & V(x) \\ 0 & V(x) & W(x) \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} a(x) & b(x)^{\mathsf{T}} & 0 \\ b(x) & H(x) + 2\varepsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0, \forall x \in \mathcal{K}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & U(x) - H(x) - 2\varepsilon I & V(x) \\ 0 & V(x)^{\mathsf{T}} & W(x) \end{bmatrix}}_{\succeq 0, \forall x \in \mathcal{K}}.$$

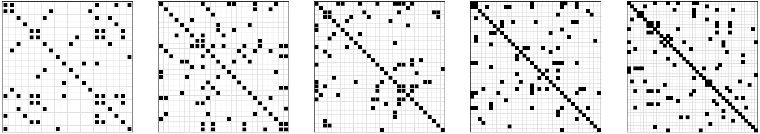
Experiment 2: PMI locally

$$B^*_{m,d} := \max_{s_{2d}(x)} \quad \int_{\mathcal{K}} s_{2d}(x)\,\mathrm{d}x$$
$$\text{subject to} \quad P(x) - s_{2d}(x) I \succeq 0 \quad \forall x \in \mathcal{K}.$$

- Set approximation: $\mathcal{P} = \{x \in \mathbb{R}^n \mid P(x) \succeq 0\} \subset \mathcal{K}$
- the unit disk: $\mathcal{K} = \{x \in \mathbb{R}^2 : 1 - x_1^2 - x_2^2 \geq 0\}$ and

$$P(x) = (1 - x_1^2 - x_2^2) I_m + (x_1 + x_1 x_2 - x_1^3) A + (2x_1^2 x_2 - x_1 x_2 - 2x_2^3) B,$$

A, B with chordal sparsity graphs, zero diagonal elements, and other entries from the uniform distribution on $(0, 1)$.



(a) $m = 20$ (b) $m = 25$ (c) $m = 30$ (d) $m = 35$ (e) $m = 40$

Figure: Chordal sparsity patterns for the polynomial matrix $P(x)$.

Experiment 2: PMI locally

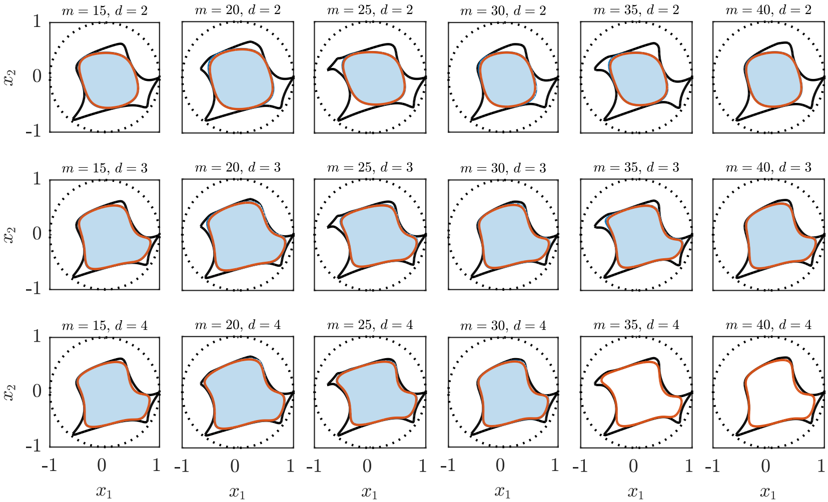


Figure: The real boundary of $\mathcal{P}$: a solid black line. Standard SOS: blue solid boundary and blue shading; the sparsity-exploiting SOS: red solid boundary, no shading.

Table: Lower bounds and CPU time (seconds, by Mosek) using the standard SOS and the sparsity-exploiting SOS. The asymptotic value $B^*_{m,\infty}$ was found by integrating the minimum eigenvalue function of P over the unit disk $\mathcal{K}$.

	Standard SOS						Sparse SOS						
	$d=2$		$d=3$		$d=4$		$d=2$		$d=3$		$d=4$		
m	t	$B^{\text{sos}}_{m,d}$	t	$B^{\text{sos}}_{m,d}$	t	$B^{\text{sos}}_{m,d}$	t	$B^{\text{sos}}_{m,d}$	t	$B^{\text{sos}}_{m,d}$	t	$B^{\text{sos}}_{m,d}$	$B^*_{m,\infty}$
15	3.7	-2.07	24.8	-1.50	95.1	-1.36	0.95	-2.10	0.97	-1.52	1.94	-1.37	-1.15
20	13.3	-1.51	96.5	-1.03	375	-0.92	0.69	-1.58	1.06	-1.07	2.12	-0.95	-0.75
25	38.1	-2.47	326	-1.85	1308	-1.64	0.95	-2.50	1.28	-1.87	3.04	-1.66	-1.41
30	136	-2.13	963	-1.54	4031	-1.41	0.75	-2.21	1.35	-1.58	3.14	-1.43	-1.21
35	219	-2.46	2210	-1.82	OOM	OOM	0.77	-2.51	1.51	-1.84	3.01	-1.65	-1.40
40	550	-2.22	5465	-1.59	OOM	OOM	1.03	-2.24	2.07	-1.59	5.62	-1.47	-1.25

Experiment 2: PMI locally

Table: Lower bounds $B^{\mathrm{sos}}_{15,d}$ on the asymptotic value $B^*_{15,\infty} = -1.153$ for $m = 15$, calculated using the sparsity-exploiting SOS with $\nu = 0$ and the standard SOS. The CPU time (t, seconds) to compute these bounds using MOSEK is also reported.

	d	6	8	10	12	14
Sparse SOS	$B^{\mathrm{sos}}_{15,d}$	-1.257	-1.219	-1.199	-1.195	-1.191
	t	13.3	85.1	309.3	818.3	2149
Standard SOS	$B^{\mathrm{sos}}_{15,d}$	-1.252	-1.216	OOM	OOM	OOM
	t	1133	8250	OOM	OOM	OOM

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