Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

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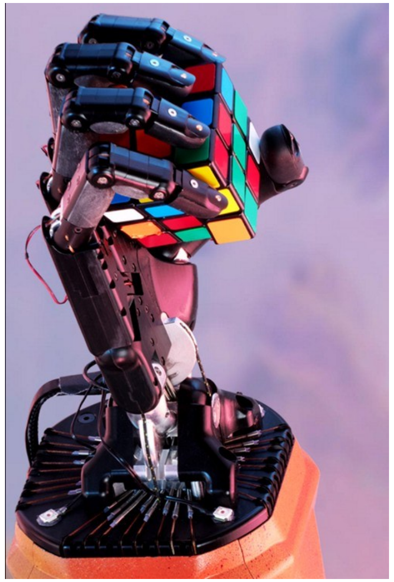
Motivation

❑ Model-free methods and data-driven control

- Use direct policy updates
- Become very popular in both academia and practice, from game playing, robotics, and drones, etc.

DeepMind



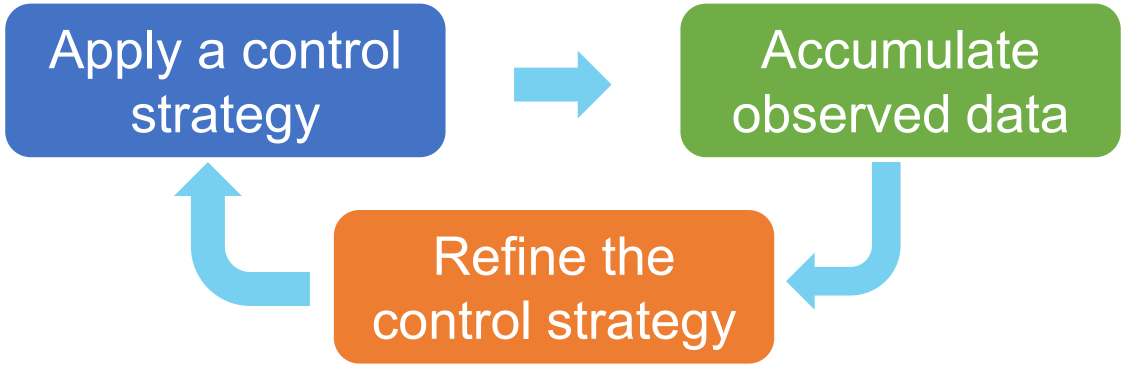
OpenAI

Applications

Duan et al. 2016; Silver et al., 2017; Dean et al., 2019; Tu and Recht, 2019;
Mania et al., 2019; Fazel et al., 2018; Recht, 2019;

Motivation

❑ Model-free methods and data-driven control



Opportunities

- Directly search over a given policy class
- Directly optimize performance on the true system, bypassing the model estimation (not on an approximated model)

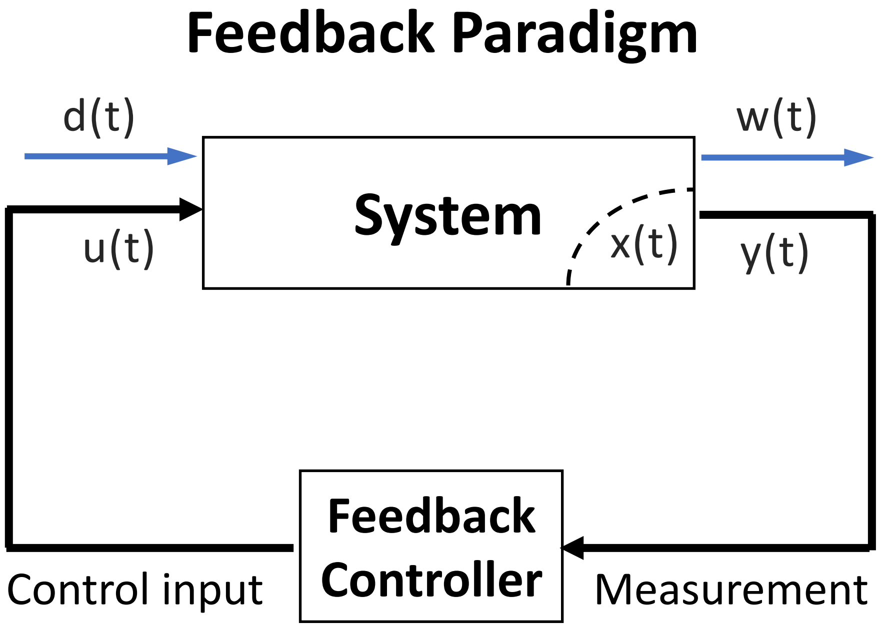
Challenges

- Lack of non-asymptotic performance guarantees
  - Sample complexity
  - Suboptimality
  - Convergence, etc.

❖ Highly nontrivial even for **linear dynamical systems**

Today's talk

❑ Optimal Control



Control theory: the principled use of feedback loops and algorithms to drive a dynamical system to its desired goal

Linear Quadratic Optimal control

$$\min_{u_1,u_2,\ldots,} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^{\mathsf{T}} Q x_t + u_t^{\mathsf{T}} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = Ax_t + Bu_t + w_t$$

$$y_t = Cx_t + v_t$$

- Many practical applications
- **Linear Quadratic Regulator (LQR)** when the state x_t is directly observable
- **Linear Quadratic Gaussian (LQG) control** when only partial output y_t is observed
- Extensive classical results (Dynamic programming, Separation principle, Riccati equations, etc.)

Major challenge: how to perform optimal control when the system is unknown?

Model-free: Direct policy iteration

❑ Controller parameterization

- Give a parameterization of control policies; say **neural networks?** ✖
- Control theory already tells us many structural properties
- **Linear feedback is sufficient for LQR** $u_t = K x_t$

$$\lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^{\mathsf{T}} Q x_t + u_t^{\mathsf{T}} R u_t\right)\right] := J(K)$$

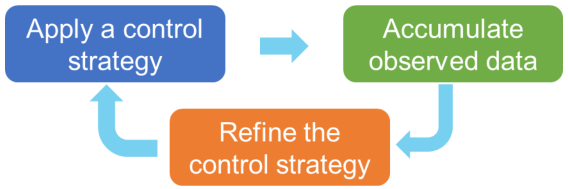
- Set of stabilizing controllers $K \in \mathcal{K}$

- A fast-growing list of references

LQR as an Optimization problem

$$\min_{K} \quad J(K)$$
$$\text{s.t.} \quad K \in \mathcal{K}$$

Direct policy iteration



$$K_{i+1} = K_i - \alpha_i \nabla J(K_i)$$

✓ Good Landscape properties (Fazel et al., 2018)

- Connected feasible region
- Unique stationary point
- Gradient dominance

✓ Fast global convergence (exponential)

➢ Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others

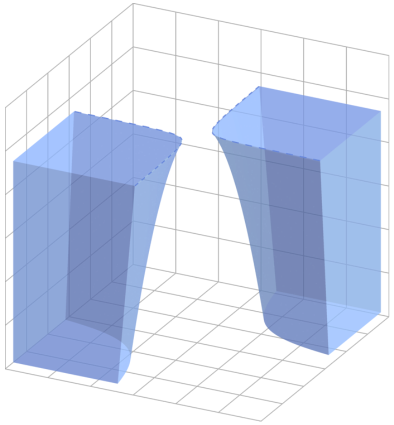
Challenges for partially observed LQG

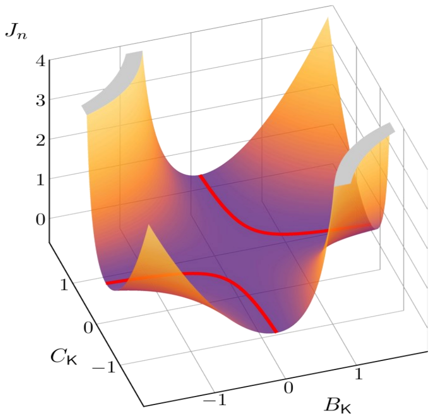
❑ Results on model-free LQG control are much fewer

- LQG is more complicated than LQR
- Requires dynamical controllers
- Its landscape properties are much richer and more complicated than LQR

<u>Our focus: Landscape Analysis of LQG</u>

- **Question 1: Properties of the domain** (set of stabilizing controllers)
  - convexity, connectivity, open/closed?
- **Question 2: Properties of the accumulated cost**
  - convexity, differentiability, coercivity?
  - set of stationary points/local minima/global minima?





Outline

- ❑ LQG problem Setup
- ❑ Connectivity of the Set of Stabilizing Controllers
- ❑ Structure of Stationary Points of the LQG cost
- ❑ Conclusions

Outline

- ❑ **LQG problem Setup**
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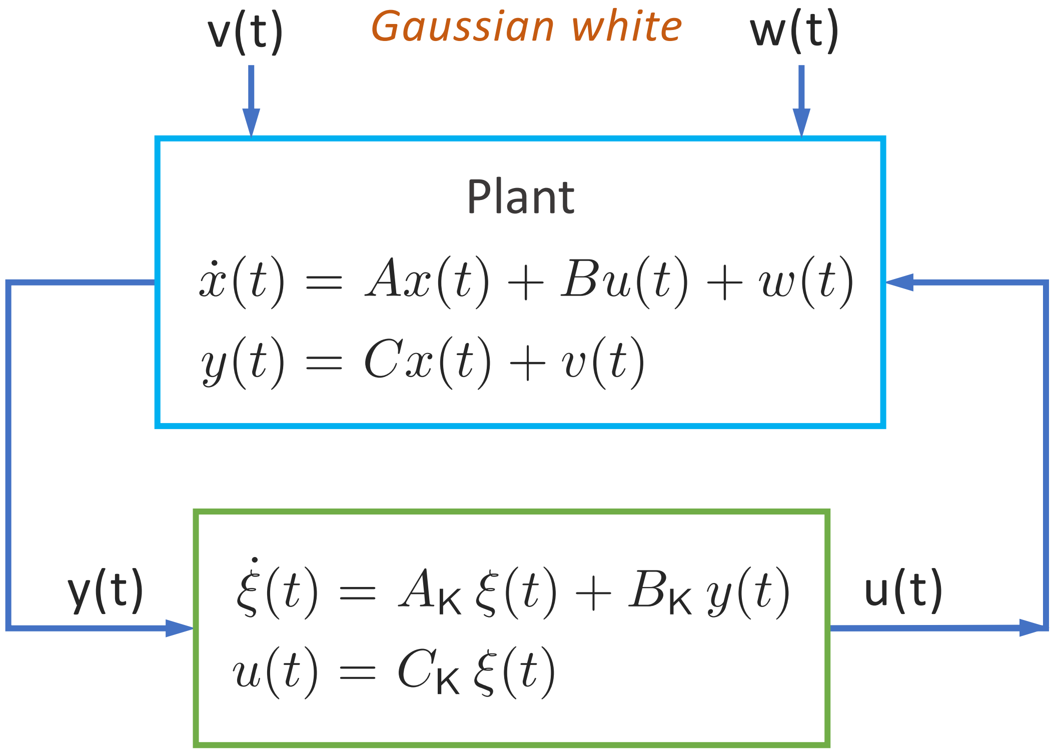
LQG Problem Setup



dynamical controller

$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$

Standard Assumption	$(A, B),\ (A, W^{1/2})$	Controllable
	$(C, A),\ (Q^{1/2}, A)$	Observable

Objective: The LQG cost

$$\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T (x^\top Q x + u^\top R u)\, dt$$

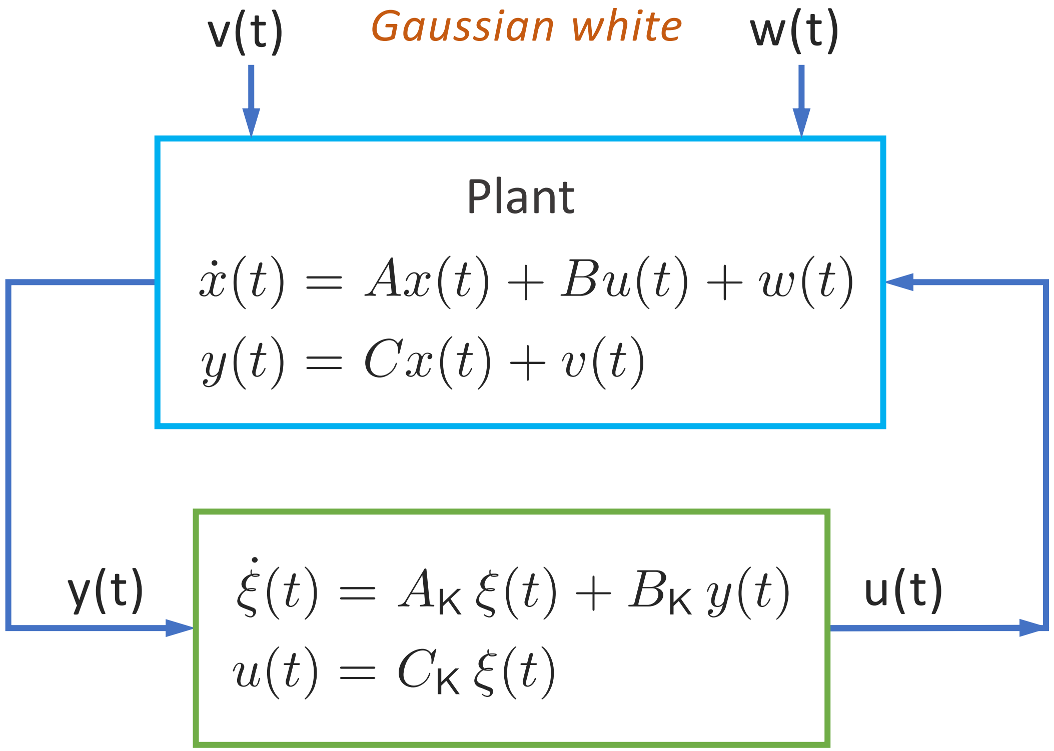
- $\xi(t)$ internal state of the controller
- $\dim \xi(t)$ order of the controller
- $\dim \xi(t) = \dim x(t)$ full-order
- $\dim \xi(t) < \dim x(t)$ reduced-order

Minimal controller

The input-output behavior cannot be replicated by a lower order controller.

* $(A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$ controllable and observable

Separation principle



dynamical controller

$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$

Explicit dependence on the dynamics

Objective: The LQG cost

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u)\, dt$$

Solution: Kalman filter for state estimation
+ LQR based on the estimated state

$$\dot{\xi} = (A - BK)\xi + L(y - C\xi),$$
$$u = -K\xi.$$

Two Riccati equations

- **Kalman gain** $L = PC^{\mathsf{T}} V^{-1}$

$$AP + PA^{\mathsf{T}} - PC^{\mathsf{T}} V^{-1} CP + W = 0,$$

- **Feedback gain** $K = R^{-1} B^{\mathsf{T}} S$

$$A^{\mathsf{T}} S + SA - SBR^{-1} B^{\mathsf{T}} S + Q = 0$$

Model-free Optimization formulation

❑ Closed-loop dynamics

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \boxed{\begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix}}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_{\mathsf{K}} \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C_{\mathsf{K}} \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}$$

❑ Feasible region of the controller parameters

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} \,\middle|\, \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \text{ is full-order, } \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \text{ is Hurwitz stable} \right\}$$

❑ Cost function $\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T (x^\top Q x + u^\top R u)\, dt$

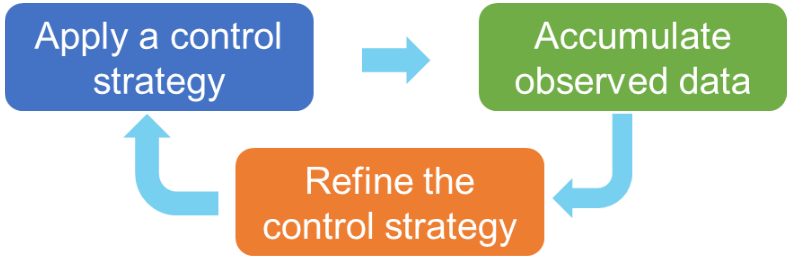
$$J(\mathsf{K}) = \operatorname{tr}\left(\begin{bmatrix} Q & 0 \\ 0 & C_{\mathsf{K}}^{\mathsf{T}} R C_{\mathsf{K}} \end{bmatrix} X_{\mathsf{K}}\right) = \operatorname{tr}\left(\begin{bmatrix} W & 0 \\ 0 & B_{\mathsf{K}} V B_{\mathsf{K}}^{\mathsf{T}} \end{bmatrix} Y_{\mathsf{K}}\right)$$

$X_{\mathsf{K}}, Y_{\mathsf{K}}$ Solution to Lyapunov equations

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$
$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

Direct policy iteration $\mathsf{K}_{i+1} = \mathsf{K}_i - \alpha_i \nabla J(\mathsf{K}_i)$

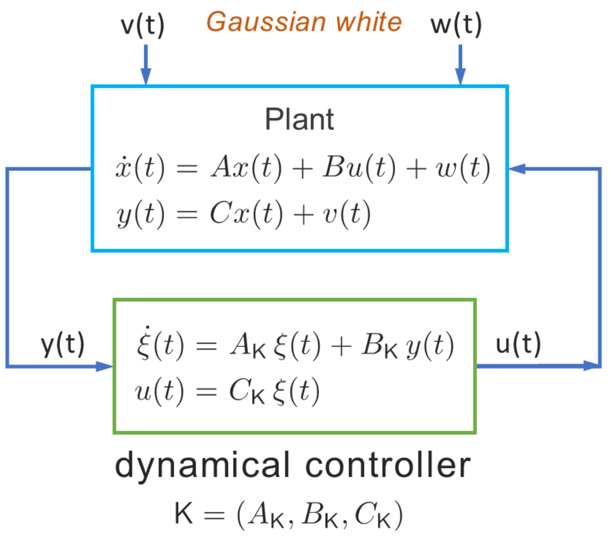


- ✓ **Does it converge at all?**
- ✓ **Converge to which point?**
- ✓ **Convergence speed?**

Landscape Analysis

Hyland, David, and Dennis Bernstein. "The optimal projection equations for fixed-order dynamic compensation." *IEEE Transactions on Automatic Control* 29.11 (1984): 1034-1037.

Model-free Optimization formulation



Landscape Analysis

LQG as an Optimization problem

$$\begin{aligned} \min_{\mathsf{K}} \quad & J(\mathsf{K}) \\ \text{s.t.} \quad & \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}} \end{aligned}$$

- **Q1: Connectivity of the feasible region $\mathcal{C}_{\text{full}}$**
  - Is it connected?
  - If not, how many connected components can it have?
- **Q2: Structure of stationary points of $J(\mathsf{K})$**
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?

Outline

- ❑ LQG problem Setup
- ❑ **Connectivity of the Set of Stabilizing Controllers**
- ❑ Structure of Stationary Points of the LQG cost
- ❑ Conclusions

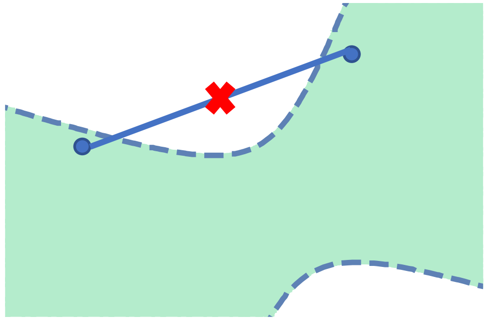
Connectivity of the feasible region

❑ Simple observation: non-convex and unbounded

Lemma 1: the set $\mathcal{C}_{\text{full}}$ is non-empty, unbounded, and can be non-convex.

Example

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$



$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, \begin{bmatrix} 1 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \text{ is stable} \right\}.$$

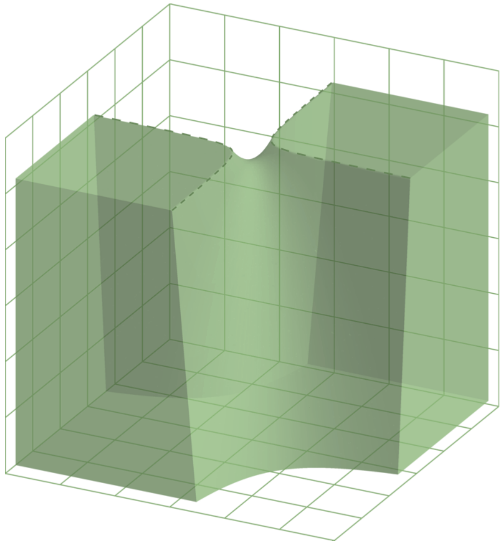
$$\mathsf{K}^{(1)} = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}, \qquad \mathsf{K}^{(2)} = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix}$$ Stabilize the plant, and thus belong to $\mathcal{C}_{\text{full}}$

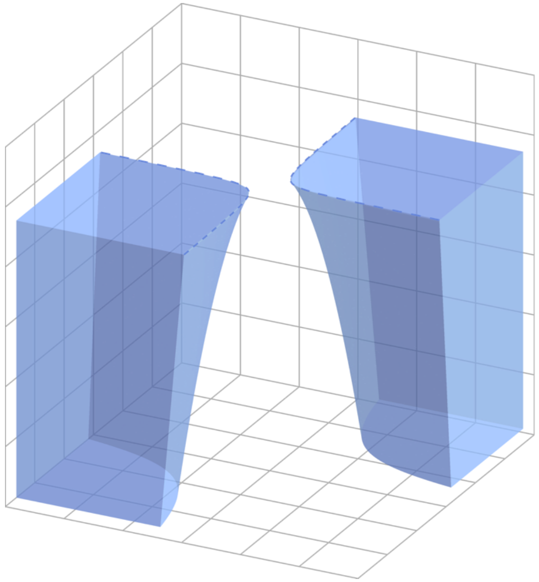
$$\hat{\mathsf{K}} = \frac{1}{2}\left(\mathsf{K}^{(1)} + \mathsf{K}^{(2)}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$ Fails to stabilize the plant, and thus outside $\mathcal{C}_{\text{full}}$

Connectivity of the feasible region

❑ Main Result 1: dis-connectivity

Theorem 1: The set $\mathcal{C}_{\text{full}}$ can be disconnected but has at most 2 connected components.



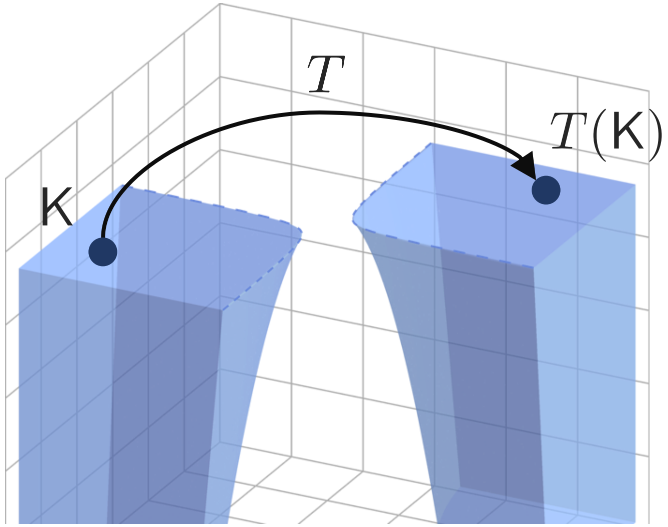


- ✓ Different from the connectivity of static stabilizing state-feedback controllers, which is always connected!
- ✓ **Is this a negative result for gradient-based algorithms?** → **No**

Connectivity of the feasible region

❑ Main Result 2: dis-connectivity

Theorem 2: If $\mathcal{C}_{\text{full}}$ has 2 connected components, then there is a smooth bijection T between the 2 connected components that has the same cost function value.



$$J(\mathsf{K}) = J(T(\mathsf{K}))$$

✓ In fact, the bijection T is defined by a similarity transformation (change of controller state coordinate)

$$\mathscr{T}_T(\mathsf{K}) := \begin{bmatrix} D_{\mathsf{K}} & C_{\mathsf{K}} T^{-1} \\ T B_{\mathsf{K}} & T A_{\mathsf{K}} T^{-1} \end{bmatrix}.$$

Positive news: For gradient-based local search methods, it makes no difference to search over either connected component.

Connectivity of the feasible region

Main Result 3: conditions for connectivity

Theorem 3: 1) $\mathcal{C}_{\text{full}}$ is connected if there exists a reduced-order stabilizing controller.

2) The sufficient condition above becomes necessary if the plant is single-input or single-output.

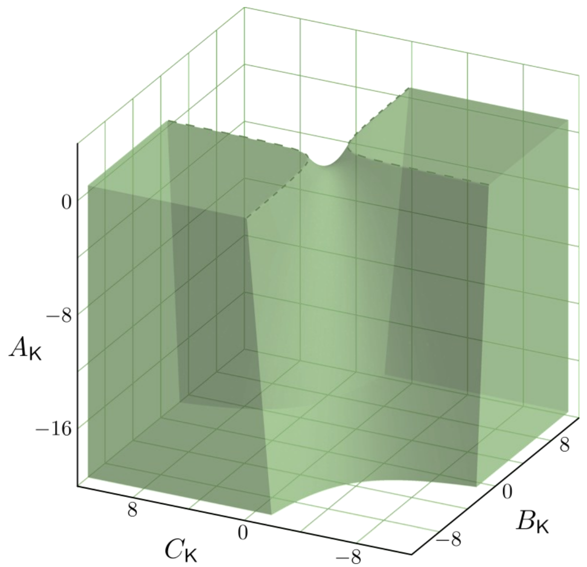
Corollary 1: Given any open-loop stable plant, the set of stabilizing controllers $\mathcal{C}_{\text{full}}$ is connected.

Example: Open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

Routh--Hurwitz stability criterion

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \middle| A_{\mathsf{K}} < 1, B_{\mathsf{K}} C_{\mathsf{K}} < -A_{\mathsf{K}} \right\}.$$



Connectivity of the feasible region

❑ Main Result 3: conditions for connectivity

Example: Open-loop unstable system (SISO)

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

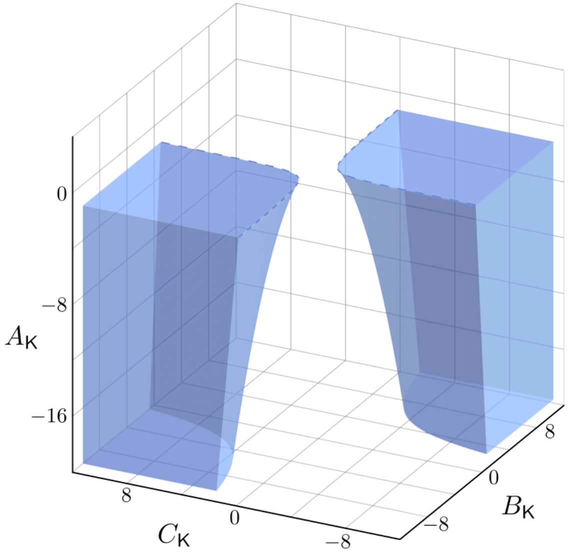
- **Routh--Hurwitz stability criterion**

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \text{ is stable} \right\}$$
$$= \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_{\mathsf{K}} < -1,\ B_{\mathsf{K}}C_{\mathsf{K}} < A_{\mathsf{K}} \right\}.$$

- **Two path-connected components**

$$\mathcal{C}_1^+ := \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_{\mathsf{K}} < -1,\, B_{\mathsf{K}}C_{\mathsf{K}} < A_{\mathsf{K}},\, B_{\mathsf{K}} > 0 \right\},$$
$$\mathcal{C}_1^- := \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_{\mathsf{K}} < -1,\, B_{\mathsf{K}}C_{\mathsf{K}} < A_{\mathsf{K}},\, B_{\mathsf{K}} < 0 \right\}.$$

Disconnected feasible region



Proof idea: Lifting via Change of Variables

❑ Change of variables in state-space domain: Lyapunov theory

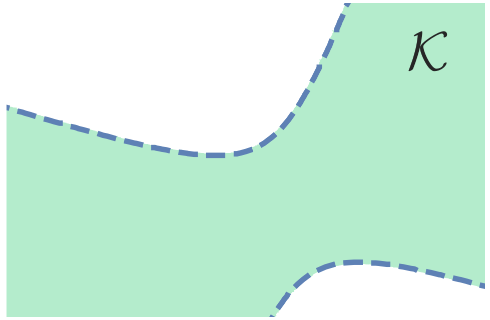
- Connectivity of the static stabilizing state feedback gains

$$\{K \in \mathbb{R}^{m\times n} \mid A - BK \text{ is stable}\}$$

$$\iff \{K \in \mathbb{R}^{m\times n} \mid \exists P \succ 0, P(A-BK)^{\mathsf{T}} + (A-BK)P \prec 0\}$$

$$\iff \{K \in \mathbb{R}^{m\times n} \mid \exists P \succ 0, PA^{\mathsf{T}} - L^{\mathsf{T}}B^{\mathsf{T}} + AP - BL \prec 0, L = KP\}$$

$$\iff \{K = LP^{-1} \in \mathbb{R}^{m\times n} \mid \exists P \succ 0, PA^{\mathsf{T}} - L^{\mathsf{T}}B^{\mathsf{T}} + AP - BL \prec 0\}.$$



Open, connected, possibly nonconvex

- How about the set of stabilizing dynamical controllers

$$\begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \text{ is stable}$$

$$\iff \quad \exists P \succ 0,\ P\begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} P \prec 0,$$

Change of variables for output feedback control is highly non-trivial

[Gahinet and Apkarian, 1994]
[Scherer et al., IEEE TAC 1997]

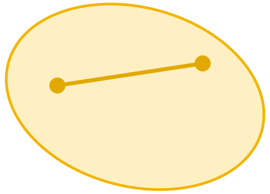
Proof idea: Lifting via Change of Variables

❑ Change of variables in state-space domain: Lyapunov theory

$$\Phi(\mathsf{Z}) = \begin{bmatrix} \Phi_D(\mathsf{Z}) & \Phi_C(\mathsf{Z}) \\ \Phi_B(\mathsf{Z}) & \Phi_A(\mathsf{Z}) \end{bmatrix} := \begin{bmatrix} I & 0 \\ YB & \Xi \end{bmatrix}^{-1} \begin{bmatrix} G & H \\ F & M - YAX \end{bmatrix} \begin{bmatrix} I & CX \\ 0 & \Pi \end{bmatrix}^{-1}.$$

[Scherer et al., IEEE TAC 1997]
[Gahinet and Apkarian, 1994]

$$\mathcal{F} \times \mathrm{GL} \xrightarrow[\text{surjective}]{\Phi} \mathcal{C}_{\text{full}}$$



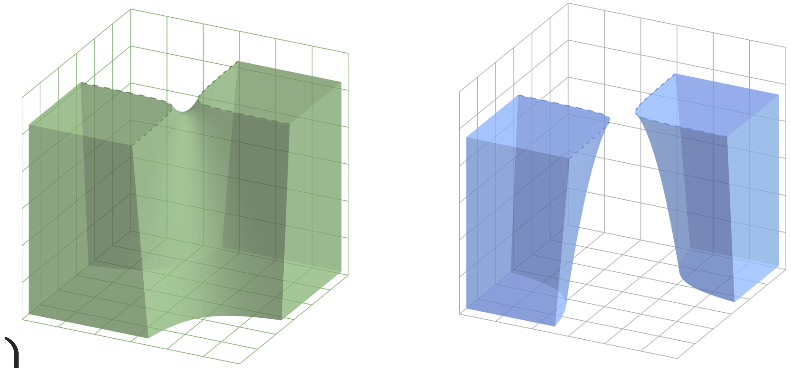
Convex thus connected

General linear group: the set of invertible matrices (similarity transformation)

at most 2 connected components

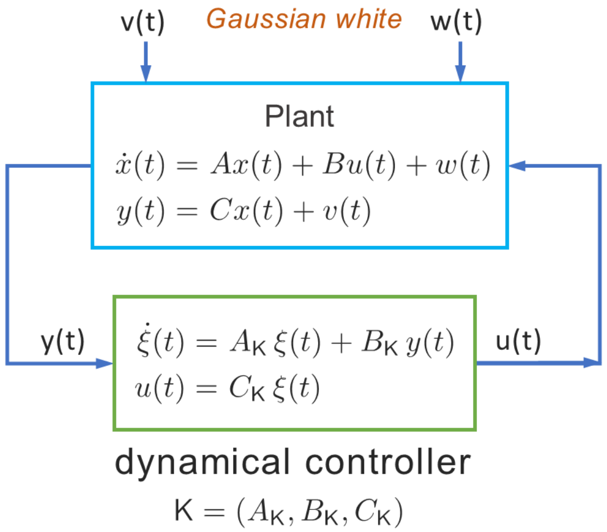
Two connected components

$$\mathrm{GL}_n^+ = \{\Pi \in \mathbb{R}^{n\times n} \mid \det \Pi > 0\},$$
$$\mathrm{GL}_n^- = \{\Pi \in \mathbb{R}^{n\times n} \mid \det \Pi < 0\}.$$



$$\mathcal{F} = \Bigg\{(X, Y, M, H, F) | X, Y \in \mathbb{S}^n, M \in \mathbb{R}^{n\times n}, H \in \mathbb{R}^{n\times p}, F \in \mathbb{R}^{m\times n},$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0, \begin{bmatrix} AX+BF & A \\ M & YA+HC \end{bmatrix} + \begin{bmatrix} AX+BF & A \\ M & YA+HC \end{bmatrix}^{\top} \prec 0 \Bigg\}$$

Model-free Optimization formulation



Landscape Analysis

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$
$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

- Q1: Connectivity of the feasible region $\mathcal{C}_{\text{full}}$
  - Is it connected? **No**
  - How many connected components can it have? **Two**
- Q2: Structure of stationary points of $J(\mathsf{K})$
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?

Outline

- ❑ LQG problem Setup
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- ❑ **Structure of Stationary Points of the LQG cost**
- ❑ Conclusions

Structure of Stationary Points

❑ Simple observations

1) $J(\mathsf{K})$ is a real analytic function over its domain (smooth, infinitely differentiable)
2) $J(\mathsf{K})$ has **non-unique** and **non-isolated** global optima

LQG as an Optimization problem

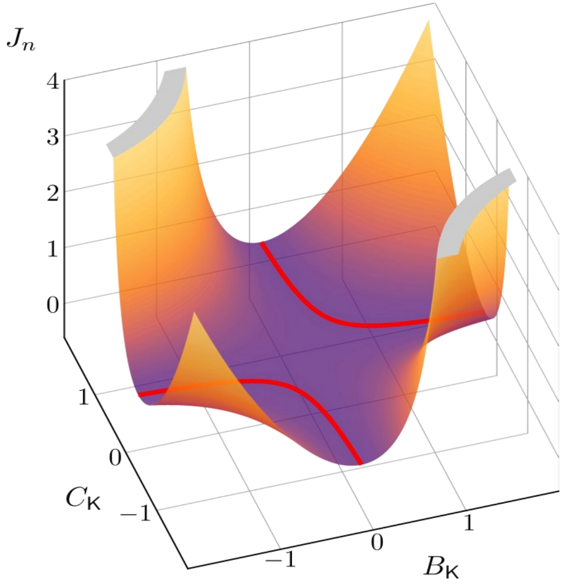
$$\begin{aligned} \min_{\mathsf{K}} \quad & J(\mathsf{K}) \\ \text{s.t.} \quad & \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}} \end{aligned}$$

$$\begin{aligned} \dot{\xi}(t) &= A_{\mathsf{K}}\, \xi(t) + B_{\mathsf{K}}\, y(t) \\ u(t) &= C_{\mathsf{K}}\, \xi(t) \end{aligned}$$

Similarity transformation

$$(A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \mapsto (TA_{\mathsf{K}}T^{-1}, TB_{\mathsf{K}}, C_{\mathsf{K}}T^{-1})$$

- $J(\mathsf{K})$ is invariant under similarity transformations.
- It has many stationary points, unlike the LQR with a unique stationary point



Structure of Stationary Points

❑ Gradient computation

Lemma 1: For every $\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$, we have

$$\frac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 2\left(Y_{12}^{\mathsf{T}} X_{12} + Y_{22} X_{22}\right),$$
$$\frac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 2\left(Y_{22} B_{\mathsf{K}} V + Y_{22} X_{12}^{\mathsf{T}} C^{\mathsf{T}} + Y_{12}^{\mathsf{T}} X_{11} C^{\mathsf{T}}\right),$$
$$\frac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 2\left(R C_{\mathsf{K}} X_{22} + B^{\mathsf{T}} Y_{11} X_{12} + B^{\mathsf{T}} Y_{12} X_{22}\right),$$

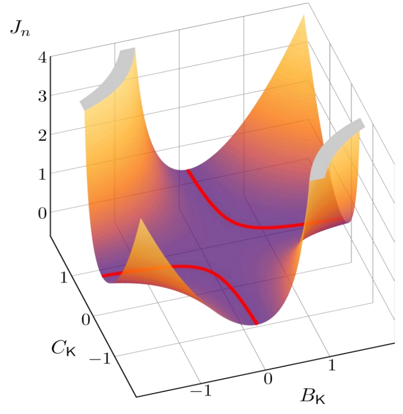
where $X_{\mathsf{K}} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix}, \qquad Y_{\mathsf{K}} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathsf{T}} & Y_{22} \end{bmatrix}$

are the unique positive semidefinite solutions to two Lyapunov equations.

How does the set of Stationary Points look like?

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \frac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 0, \; \frac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 0, \; \frac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 0, \right\}$$

- ❑ Non-unique, non-isolated
- ❑ Local minimum, local maximum, saddle points, or globally minimum?



Structure of Stationary Points

Main Result: existences of strict saddle points

Theorem 4: Consider any open-loop stable plant. The zero controller with any stable A_{K}

$$\mathsf{K} = (A_{\mathsf{K}}, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero.

Example:

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

$$Q = 1, R = 1, V = 1, W = 1$$

Stationary point: $\mathsf{K}^\star = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad \text{with } a < 0$

- **Cost function:** $$J\left(\begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix}\right) = \frac{A_{\mathsf{K}}^2 - A_{\mathsf{K}}(1 + B_{\mathsf{K}}^2 C_{\mathsf{K}}^2) - B_{\mathsf{K}} C_{\mathsf{K}}(1 - 3B_{\mathsf{K}} C_{\mathsf{K}} + B_{\mathsf{K}}^2 C_{\mathsf{K}}^2)}{2(-1 + A_{\mathsf{K}})(A_{\mathsf{K}} + B_{\mathsf{K}} C_{\mathsf{K}})}.$$

- **Hessian:** $$\left.\begin{bmatrix} \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}}^2} & \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}} \partial B_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}} \partial C_{\mathsf{K}}} \\ \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}} A_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}}^2} & \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}} \partial C_{\mathsf{K}}} \\ \frac{\partial J^2(\mathsf{K})}{\partial C_{\mathsf{K}} A_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial C_{\mathsf{K}} B_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial \partial C_{\mathsf{K}}^2} \end{bmatrix}\right|_{\mathsf{K}^\star = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}} = \frac{1}{2(1-a)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

Indefinite with eigenvalues:

$0 \text{ and } \pm\frac{1}{2(1-a)}$

Structure of Stationary Points

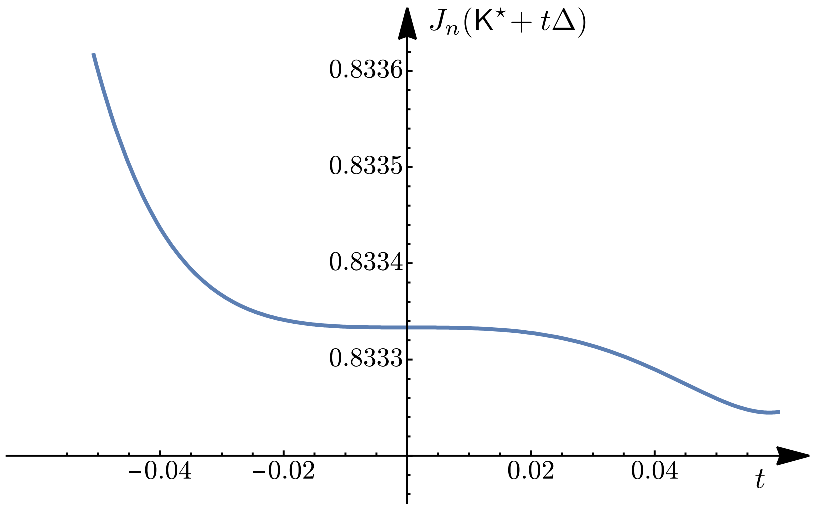
❑ Main Result: existences of strict saddle points

Theorem 4: Consider any open-loop stable plant. The zero controller with any stable A_{K}

$$\mathsf{K} = (A_{\mathsf{K}}, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero.

Another example with zero Hessian



How does the set of Stationary Points look like?

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \begin{aligned} \frac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} &= 0, \\ \frac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} &= 0, \\ \frac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} &= 0, \end{aligned} \right\}$$

- ❑ **Non-unique, non-isolated**
- ❑ **Strictly suboptimal points; Strict saddle points**
- ❑ **All bad stationary points correspond to non-minimal controllers**

Structure of Stationary Points

❑ Main Result

Theorem 5: **All stationary points corresponding to controllable and observable controllers are globally optimum.**

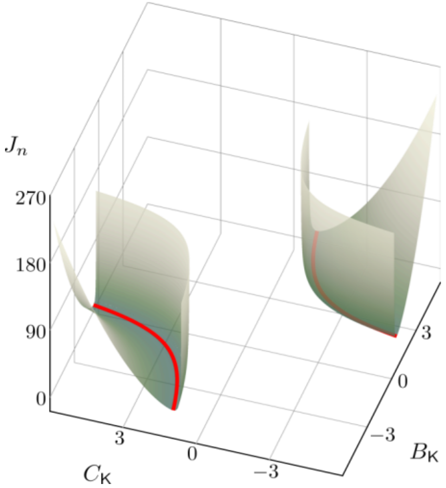
$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \begin{array}{l} \dfrac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 0, \\ \dfrac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 0, \\ \dfrac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 0, \end{array} \right\}$$

Particularly, given a stationary point that is a **minimal** controller

1) This stationary point is a global optimum of $J(\mathsf{K})$
2) The set of all global optima forms a manifold with 2 connected components. They are connected by a similarity transformation.

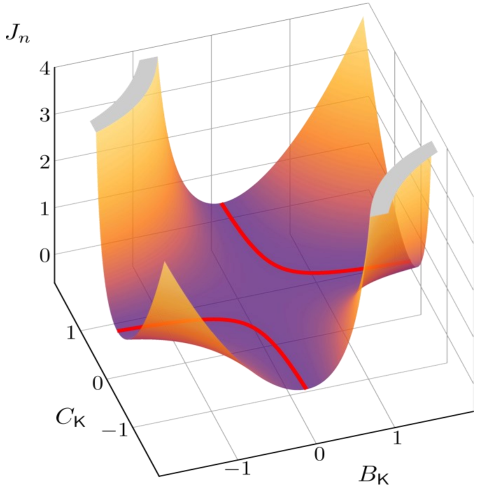
Example: open-loop unstable system

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$



Example: open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$



Proof idea

❑ Proof: all minimal stationary points are unique up to a similarity transformation

All minimal stationary points $\mathsf{K} = (A_\mathsf{K}, B_\mathsf{K}, C_\mathsf{K}) \in \mathcal{C}_{\text{full}}$ to the LQG problem are in the form of

$$A_\mathsf{K} = T(A - BK - LC)T^{-1}, \qquad B_\mathsf{K} = -TL, \qquad C_\mathsf{K} = KT^{-1},$$

$$K = R^{-1}B^\mathsf{T}S,\ L = PC^\mathsf{T}V^{-1},$$

T is an invertible matrix and P, S are the unique positive definite solutions to the Riccati equations

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0,\ \frac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 0,\ \frac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 0, \right\}$$

Minimal controller

$$\frac{\partial J_n(\mathsf{K})}{\partial B_\mathsf{K}} = 0 \quad \Longrightarrow \quad B_\mathsf{K} = -TPC^\mathsf{T}V^{-1}$$

$$\frac{\partial J_n(\mathsf{K})}{\partial C_\mathsf{K}} = 0 \quad \Longrightarrow \quad C_\mathsf{K} = R^{-1}B^\mathsf{T}ST^{-1}$$

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0 \quad \Longrightarrow \quad A_\mathsf{K} = T(A - PC^\mathsf{T}V^{-1}C - BR^{-1}B^\mathsf{T}S)T^{-1}$$

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 2\left(Y_{12}^\mathsf{T}X_{12} + Y_{22}X_{22}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 2\left(Y_{22}B_\mathsf{K}V + Y_{22}X_{12}^\mathsf{T}C^\mathsf{T} + Y_{12}^\mathsf{T}X_{11}C^\mathsf{T}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 2\left(RC_\mathsf{K}X_{22} + B^\mathsf{T}Y_{11}X_{12} + B^\mathsf{T}Y_{12}X_{22}\right),$$

Special case in Theorem 20.6 of Zhou et al., 1996 and Section II of Hyland, 1984

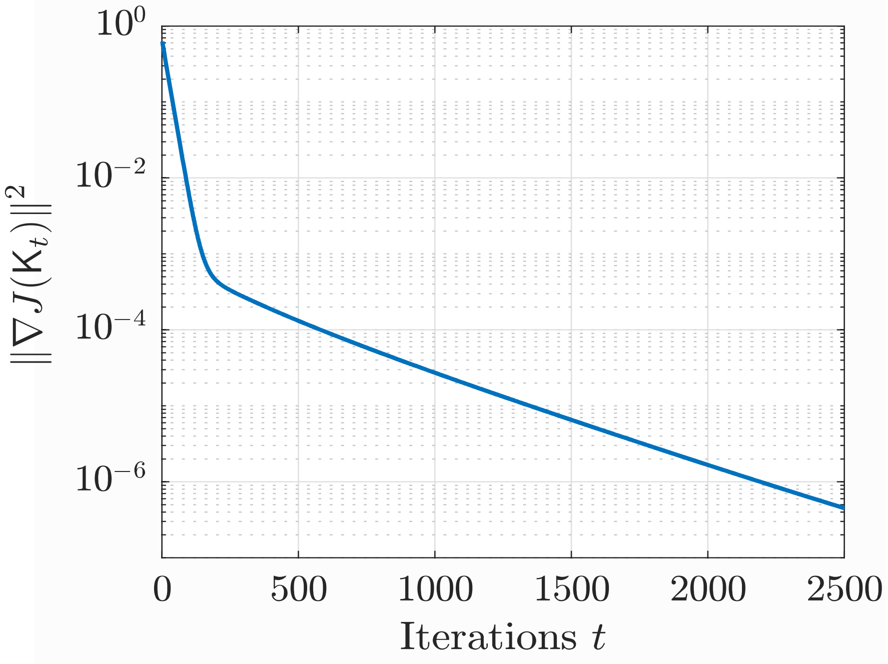
Structure of Stationary Points

❑ Implication

Corollary: Consider gradient descent iterations

$$\mathsf{K}_{t+1} = \mathsf{K}_t - \alpha \nabla J(\mathsf{K}_t)$$

If the iterates converge to a minimal controller, then this minimal controller is a global optima.



Open questions:

- ✓ Convergence conditions?
- ✓ Convergence speed?
- ✓ Alternative model-free parameterization

A very recent and related paper on output estimation is

Umenberger, J., Simchowitz, M., Perdomo, J. C., Zhang, K., & Tedrake, R. (2022). Globally Convergent Policy Search over Dynamic Filters for Output Estimation. *arXiv preprint arXiv:2202.11659*.

Comparison with LQR

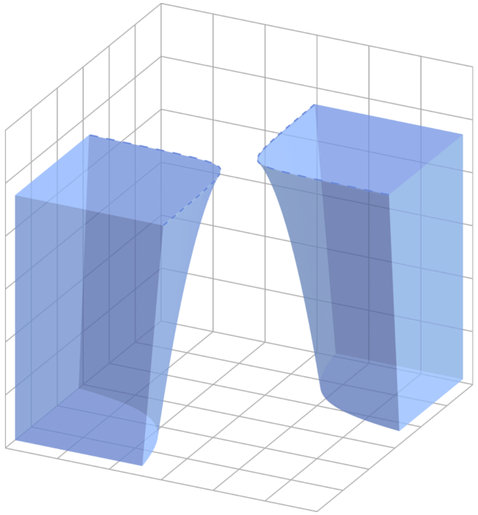
	LQR as an Optimization problem $\min_{K} \; J(K)$ s.t. $K \in \mathcal{K}$	**LQG as an Optimization problem** $\min_{\mathsf{K}} \; J(\mathsf{K})$ s.t. $\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$
Connectivity of feasible region	❖ Always connected	❖ Disconnected, but at most 2 connected comp. ❖ They are almost identical to each other
Stationary points	❖ Unique	❖ Non-unique, non-isolated stationary points ❖ Spurious stationary points (strict saddle, nonminimal controller) ❖ **All mini. stationary points are globally optimal**
Gradient Descent	❖ Gradient dominance ❖ Global fast convergence (like strictly convex)	❖ No gradient dominance ❖ Local convergence/speed (**unknown**) ❖ **Many open questions**
References	Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others	Zheng*, Tang*, Li. 2021, link (* equal contribution)

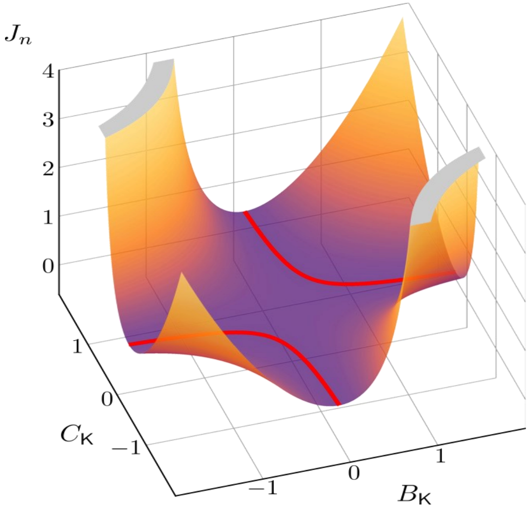
Outline

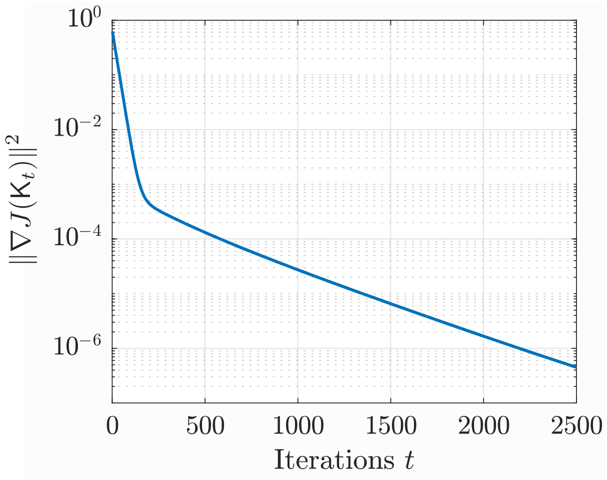
- ❑ LQG problem Setup
- ❑ Connectivity of the Set of Stabilizing Controllers
- ❑ Structure of Stationary Points of the LQG cost
- ❑ **Conclusions**

Landscape Analysis of LQG control

- Much richer and more complicated than LQR
- Disconnected, but at most 2 connected components
- Non-unique, non-isolated stationary points, strict saddle points
- Minimal stationary points are globally optimal







Ongoing and Future work

- A comprehensive classification of stationary points
- Conditions for existence of minimal globally optimal controllers
- Saddle points with vanishing Hessians may exist. How to deal with them?
- Alternative model-free parametrization of dynamic controllers
  - ✓ Better optimization landscape structures, smaller dimension
- Perturbed policy gradient?

Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

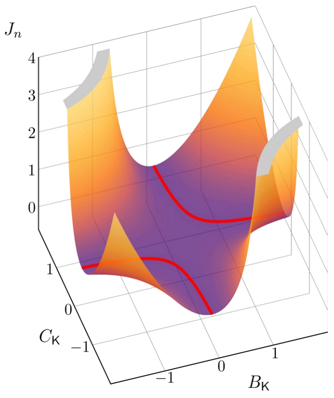
Thank you for your attention!

Q & A

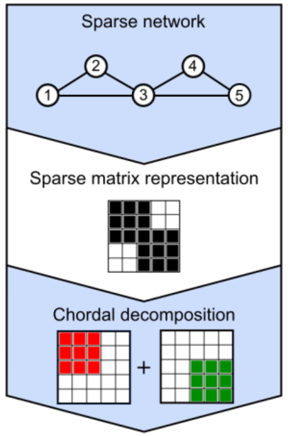
More details. Check out our paper: https://arxiv.org/abs/2102.04393

SOC lab at UC San Diego

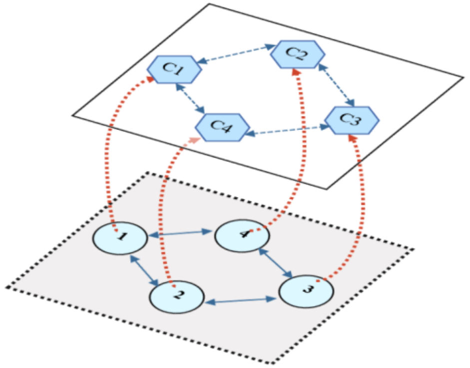
Data-driven and learning-based control



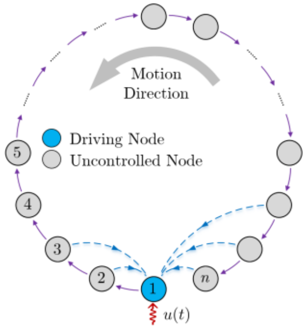
Sparse conic optimization



Scalable distributed control



Connected and autonomous vehicles (CAVs)



Check out our webpage: https://zhengy09.github.io/soclab.html

Extra Slides

Strict saddle points

Theorem 1. *Suppose the plant is open-loop stable. Let $\Lambda \in \mathbb{R}^{n\times n}$ be stable, and let*

$$\mathsf{K}^\star = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix}.$$

Then $\mathsf{K}^\star$ is a stationary point of $J_n(\mathsf{K})$ over $\mathsf{K} \in \mathcal{C}_n$, and the corresponding Hessian $\mathrm{Hess}_{\mathsf{K}^\star}$ is either indefinite or zero.

Furthermore, suppose Λ is diagonalizable, and let $\mathrm{eig}(-\Lambda)$ denote the set of (distinct) eigenvalues of $-\Lambda$. Let X_{op} and Y_{op} be the solutions to the following Lyapunov equations

$$AX_{\mathrm{op}} + X_{\mathrm{op}}A^{\mathsf{T}} + W = 0, \quad A^{\mathsf{T}}Y_{\mathrm{op}} + Y_{\mathrm{op}}A + Q = 0, \tag{1}$$

and let

$$\mathcal{Z} = \left\{ s \in \mathbb{C} \mid CX_{\mathrm{op}}\left(sI - A^{\mathsf{T}}\right)^{-1} Y_{\mathrm{op}}B = 0 \right\}. \tag{2}$$

Then, the Hessian of J_n at $\mathsf{K}^\star$ is indefinite if and only if $\mathrm{eig}(-\Lambda) \not\subseteq \mathcal{Z}$; the Hessian of J_n at $\mathsf{K}^\star$ is zero if and only if $\mathrm{eig}(-\Lambda) \subseteq \mathcal{Z}$.

Strict saddle points

Example 1. *Consider the following SISO system:*

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 11 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = 1,$$

and let

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1.$$

It can be checked that

$$C X_{\mathrm{op}} \left(sI - A^{\mathsf{T}}\right)^{-1} Y_{\mathrm{op}} B = \frac{5(s-1)}{36(s+1)(s+2)}.$$

The point

$$\mathsf{K}^{\star} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is a stationary point of J_n with a vanishing Hessian.