Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

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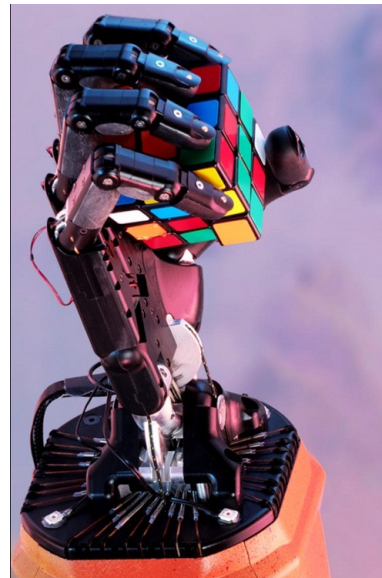
Yujie Tang
Harvard University

Na Li
Harvard University

Motivation

❑ **Model-free methods and data-driven control**

- Use direct policy updates

- Become very popular in both academia and practice, from game playing, robotics, and drones, etc.
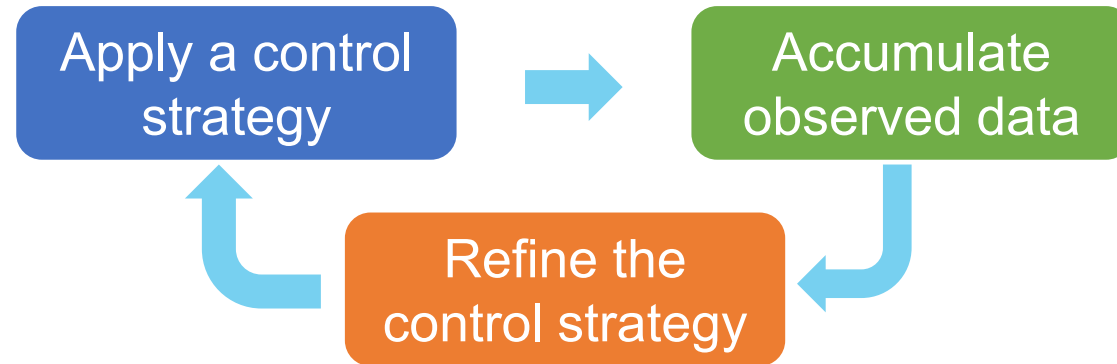
DeepMind

OpenAI

Applications

Duan et al. 2016; Silver et al., 2017; Dean et al., 2019; Tu and Recht, 2019;
Mania et al., 2019; Fazel et al., 2018; Recht, 2019;

Motivation

❑ **Model-free methods and data-driven control**

Opportunities

- Directly search over a given policy class
- Directly optimize performance on the true system, bypassing the model estimation (not on an approximated model)
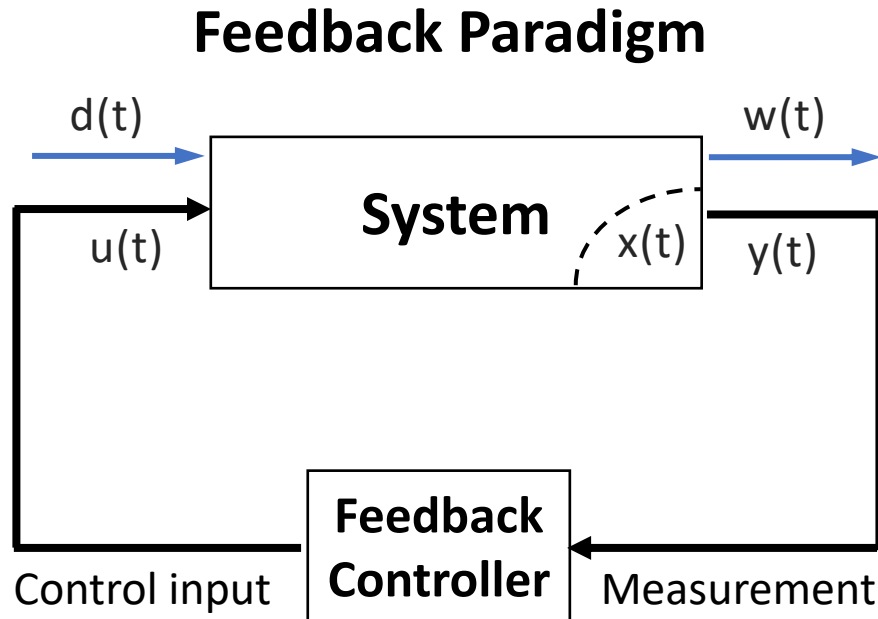
Challenges

- Lack of non-asymptotic performance guarantees
  - ➢ Sample complexity
  - ➢ Suboptimality
  - ➢ Convergence, etc.

❖ Highly nontrivial even for **linear dynamical systems**

Today's talk

□ **Optimal Control**

Feedback Paradigm

Control theory: the principled use of feedback loops and algorithms to drive a dynamical system to its desired goal

Linear Quadratic Optimal control

$$\min_{u_1, u_2, \ldots,} \quad \lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T} \sum_{t=1}^{T} \left(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = A x_t + B u_t + w_t$$

$$y_t = C x_t + v_t$$

- Many practical applications
- **Linear Quadratic Regulator (LQR)** when the state x_t is directly observable
- **Linear Quadratic Gaussian (LQG) control** when only partial output y_t is observed
- Extensive classical results (Dynamic programming, Separation principle, Riccati equations, etc.)

Major challenge: how to perform optimal control when the system is unknown?

Model-free: Direct policy iteration

❑ **Controller parameterization**

- Give a parameterization of control policies; say **neural networks?** ❌

- Control theory already tells us many structural properties

- **Linear feedback is sufficient for LQR** $u_t = Kx_t$

$$\lim_{T \to \infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^\top Q x_t + u_t^\top R u_t\right)\right] := J(K)$$

- Set of stabilizing controllers $K \in \mathcal{K}$

- A fast-growing list of references

LQR as an Optimization problem

$$\min_{K} \quad J(K)$$

$$\text{s.t.} \quad K \in \mathcal{K}$$

Direct policy iteration

$$K_{i+1} = K_i - \alpha_i \nabla J(K_i)$$

Apply a control strategy → Accumulate observed data → Refine the control strategy

- ✓ Good Landscape properties (Fazel et al., 2018)
  - Connected feasible region
  - Unique stationary point
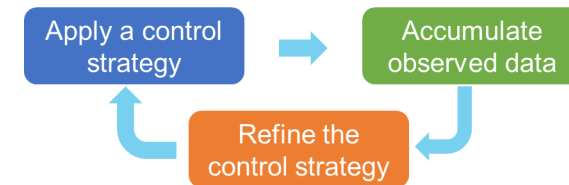 - Gradient dominance
- ✓ Fast global convergence (exponential)

➤ Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others

Challenges for partially observed LQG

❑ **Results on model-free LQG control are much fewer**

- LQG is more complicated than LQR

- Requires dynamical controllers

- Its landscape properties are much richer and more complicated than LQR

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Our focus: Landscape Analysis of LQG

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▪ **Question 1: Properties of the domain** (set of stabilizing controllers)

- convexity, connectivity, open/closed?

▪ **Question 2: Properties of the accumulated cost**

- convexity, differentiability, coercivity?
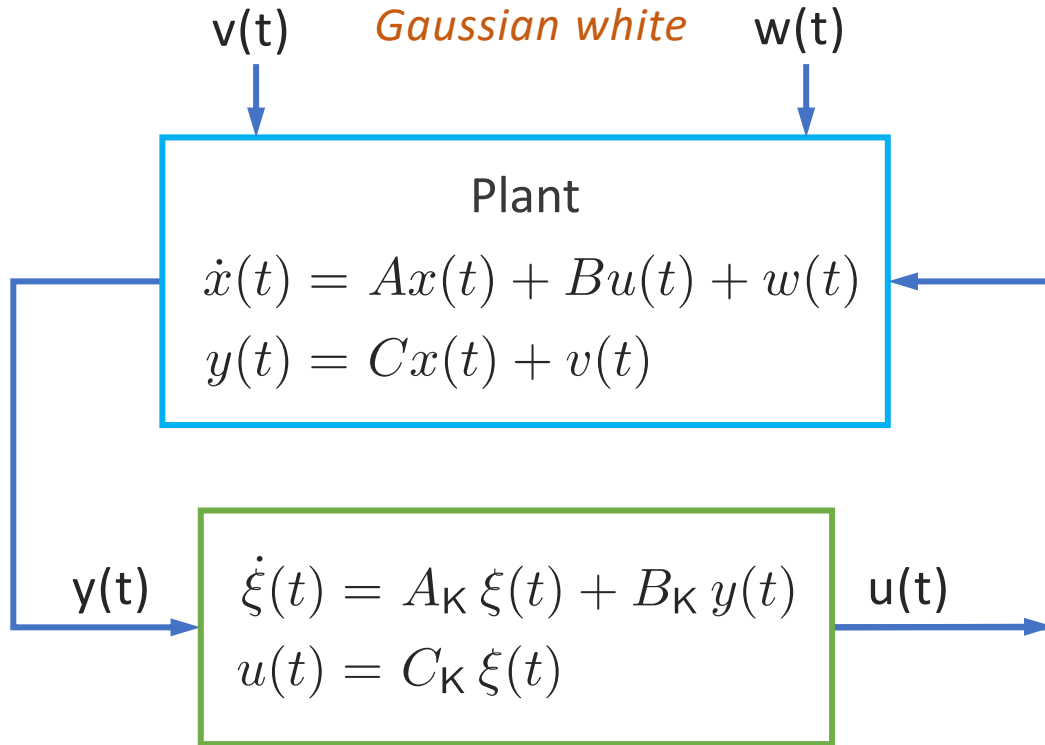
- set of stationary points/local minima/global minima?
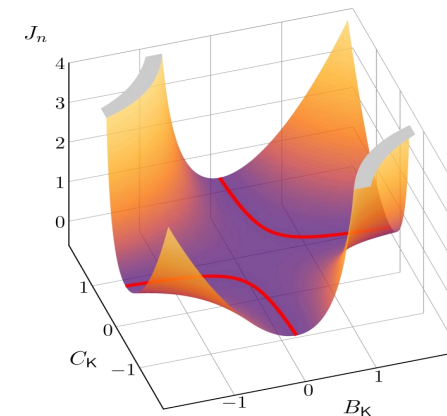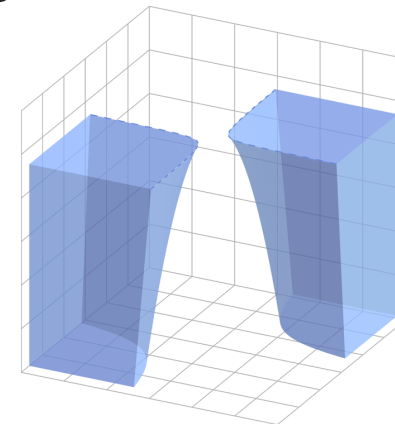
Outline

❑ **LQG problem Setup**

❑ **Connectivity of the Set of Stabilizing Controllers**

❑ **Structure of Stationary Points of the LQG cost**

❑ **Conclusions**

Outline

☐ **LQG problem Setup**

☐ Connectivity of the Set of Stabilizing Controllers

☐ Structure of Stationary Points of the LQG cost

☐ Conclusions

LQG Problem Setup

v(t)   *Gaussian white*   w(t)

Plant

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = Cx(t) + v(t)$$

y(t)

$$\dot{\xi}(t) = A_{\mathsf{K}}\,\xi(t) + B_{\mathsf{K}}\,y(t)$$
$$u(t) = C_{\mathsf{K}}\,\xi(t)$$

u(t)

dynamical controller

$$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$$

Standard Assumption

$(A, B), (A, W^{1/2})$ Controllable

$(C, A), (Q^{1/2}, A)$ Observable

Objective: The LQG cost

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u)\, dt$$
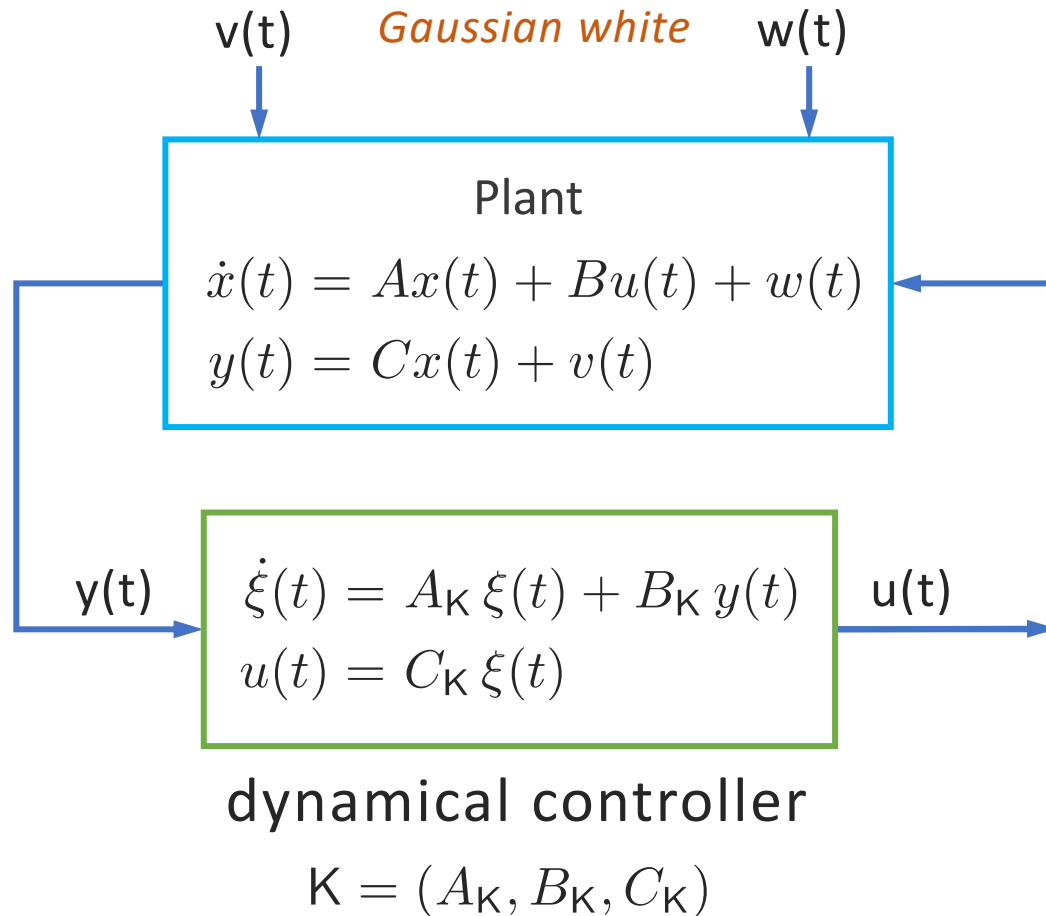
➤ $\xi(t)$ internal state of the controller

➤ $\dim \xi(t)$ order of the controller

➤ $\dim \xi(t) = \dim x(t)$ full-order

➤ $\dim \xi(t) < \dim x(t)$ reduced-order

Minimal controller

The input-output behavior cannot be replicated by a lower order controller.

$*(A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$ controllable and observable

Separation principle

v(t)   *Gaussian white*   w(t)

Plant

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = Cx(t) + v(t)$$

y(t)

$$\dot{\xi}(t) = A_{\mathsf{K}}\,\xi(t) + B_{\mathsf{K}}\,y(t)$$
$$u(t) = C_{\mathsf{K}}\,\xi(t)$$

u(t)

dynamical controller

$$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$$

Explicit dependence on the dynamics

Objective: The LQG cost

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u)\, dt$$

Solution: Kalman filter for state estimation + LQR based on the estimated state

$$\dot{\xi} = (A - BK)\xi + L(y - C\xi),$$
$$u = -K\xi.$$

Two Riccati equations

➢ **Kalman gain** $L = PC^\mathsf{T} V^{-1}$

$$AP + PA^\mathsf{T} - PC^\mathsf{T} V^{-1} CP + W = 0,$$

➢ **Feedback gain** $K = R^{-1}B^\mathsf{T} S$

$$A^\mathsf{T} S + SA - SBR^{-1}B^\mathsf{T} S + Q = 0$$

Model-free Optimization formulation

☐ **Closed-loop dynamics**

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_K \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix},$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C_K \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}.$$

LQG as an Optimization problem

$$\min_{K} \quad J(K)$$

$$\text{s.t.} \quad K = (A_K, B_K, C_K) \in \mathcal{C}_{\text{full}}$$

☐ **Feasible region of the controller parameters**

$$\mathcal{C}_{\text{full}} = \left\{ K \,|\, K = (A_K, B_K, C_K) \text{ is full order} \right.$$

$$\left. \begin{bmatrix} A & BC_K \\ B_K C & A_K \end{bmatrix} \text{ is Hurwitz stable} \right\}$$

Direct policy iteration $\quad K_{i+1} = K_i - \alpha_i \nabla J(K_i)$

Apply a control strategy → Accumulate observed data

Refine the control strategy

✓ **Does it converge at all?**

✓ **Converge to which point?**

✓ **Convergence speed?**

Landscape Analysis

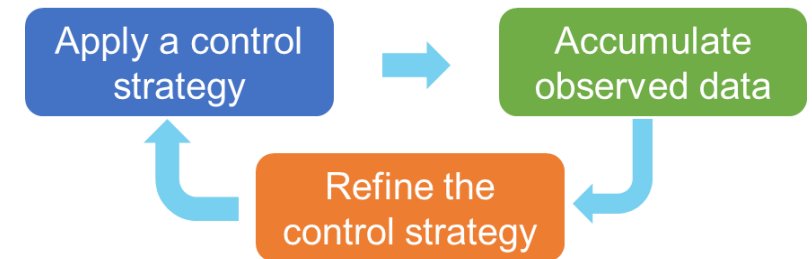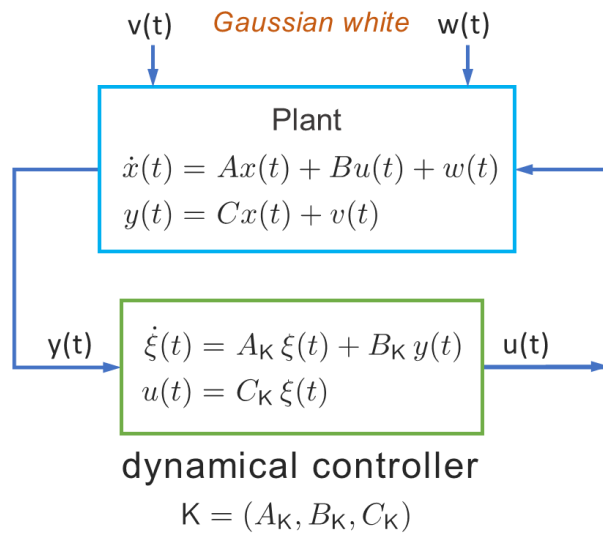☐ **Cost function** $\quad \lim_{T \to +\infty} \frac{1}{T}\mathbb{E}\int_0^T (x^\top Q x + u^\top R u)\, dt$

$$J(K) = \text{tr}\left(\begin{bmatrix} Q & 0 \\ 0 & C_K^\top R C_K \end{bmatrix} X_K\right) = \text{tr}\left(\begin{bmatrix} W & 0 \\ 0 & B_K V B_K^\top \end{bmatrix} Y_K\right)$$

X_K, Y_K Solution to Lyapunov equations

Hyland, David, and Dennis Bernstein. "The optimal projection equations for fixed-order dynamic compensation." *IEEE Transactions on Automatic Control* 29.11 (1984): 1034-1037.

Model-free Optimization formulation

v(t) *Gaussian white* w(t)

Plant
$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = Cx(t) + v(t)$$

y(t) $\dot{\xi}(t) = A_{\mathsf{K}}\,\xi(t) + B_{\mathsf{K}}\,y(t)$ u(t)
$$u(t) = C_{\mathsf{K}}\,\xi(t)$$

dynamical controller
$$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$$

Landscape Analysis

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$

$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

- **Q1: Connectivity of the feasible region $\mathcal{C}_{\text{full}}$**

  - Is it connected?

  - If not, how many connected components can it have?

- **Q2: Structure of stationary points of $J(\mathsf{K})$**

  - Are there spurious (strictly suboptimal, saddle) stationary points?

  - How to check if a stationary point is globally optimal?

Outline

Connectivity of the feasible region

❑ **Simple observation: non-convex and unbounded**

> **Lemma 1**: the set $\mathcal{C}_{\text{full}}$ is non-empty, unbounded, and can be non-convex.
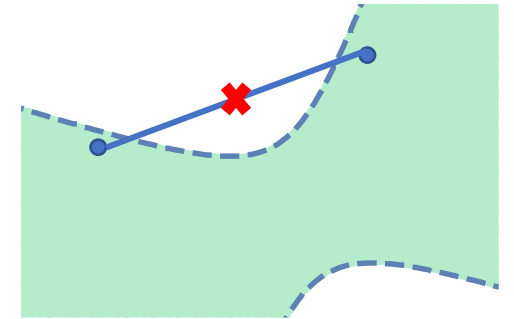
Example

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times2} \, \middle| \, \begin{bmatrix} 1 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \text{ is stable} \right\}.$$

$$\mathsf{K}^{(1)} = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}, \qquad \mathsf{K}^{(2)} = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix} \qquad \text{Stabilize the plant, and thus belong to } \mathcal{C}_{\text{full}}$$
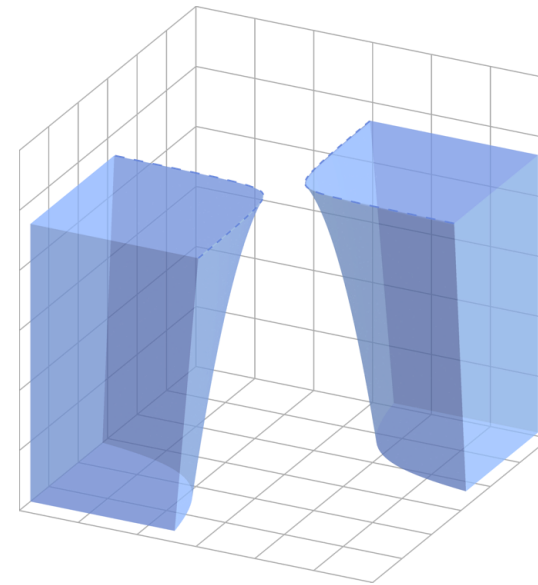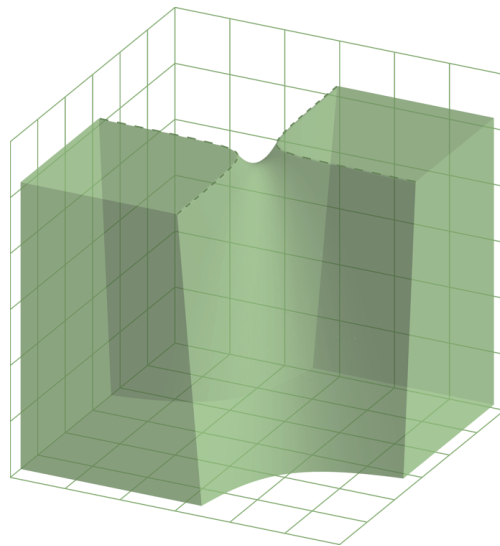
$$\hat{\mathsf{K}} = \frac{1}{2}\left(\mathsf{K}^{(1)} + \mathsf{K}^{(2)}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \qquad \text{Fails to stabilize the plant, and thus outside } \mathcal{C}_{\text{full}}$$

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Connectivity of the feasible region

❑ **Main Result 1: dis-connectivity**

**Theorem 1:** The set $\mathcal{C}_{\mathrm{full}}$ can be disconnected but has at most 2 connected components.

✓ Different from the connectivity of static stabilizing state-feedback controllers, which is always connected!

✓ **Is this a negative result for gradient-based algorithms? → No**

Connectivity of the feasible region

❑ **Main Result 2: dis-connectivity**

Theorem 2: If $\mathcal{C}_{\text{full}}$ has 2 connected components, then there is a smooth bijection T between the 2 connected components that has the same cost function value.
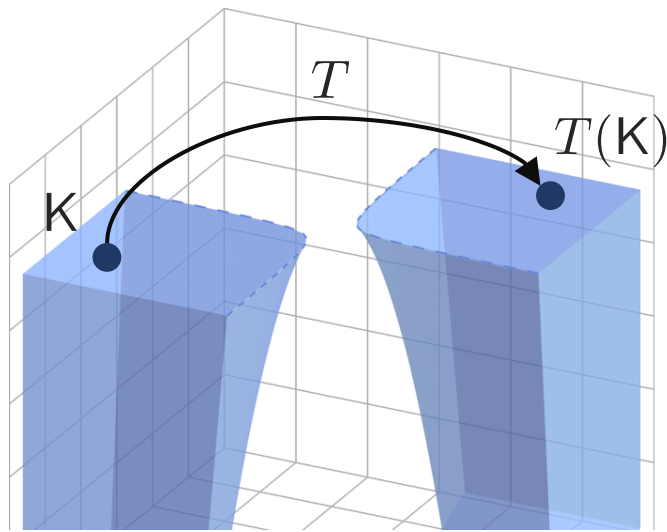
$$J(\mathsf{K}) = J(T(\mathsf{K}))$$

✓ In fact, the bijection T is defined by a similarity transformation (change of controller state coordinate)

$$\mathscr{T}_T(\mathsf{K}) := \begin{bmatrix} D_\mathsf{K} & C_\mathsf{K}T^{-1} \\ TB_\mathsf{K} & TA_\mathsf{K}T^{-1} \end{bmatrix}.$$

Positive news: For gradient-based local search methods, it makes no difference to search over either connected component.

Connectivity of the feasible region

□ **Main Result 3: conditions for connectivity**

Theorem 3: 1) $\mathcal{C}_{\text{full}}$ is connected if there exists a reduced-order stabilizing controller.

2) The sufficient condition above becomes necessary if the plant is single-input or single-output.

Corollary 1: Given any open-loop stable plant, the set of stabilizing controllers $\mathcal{C}_{\text{full}}$ is connected.
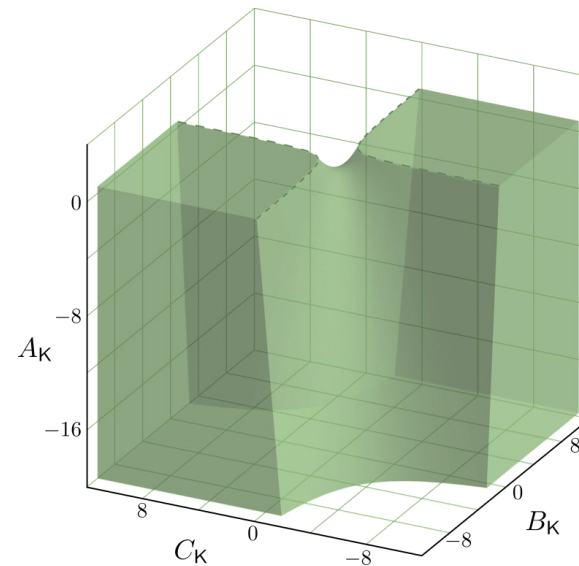
Example: Open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

Routh--Hurwitz stability criterion

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2 \times 2} \middle| A_{\mathsf{K}} < 1, B_{\mathsf{K}} C_{\mathsf{K}} < -A_{\mathsf{K}} \right\}.$$

❑ **Main Result 3: conditions for connectivity**

Example: Open-loop unstable system (SISO)
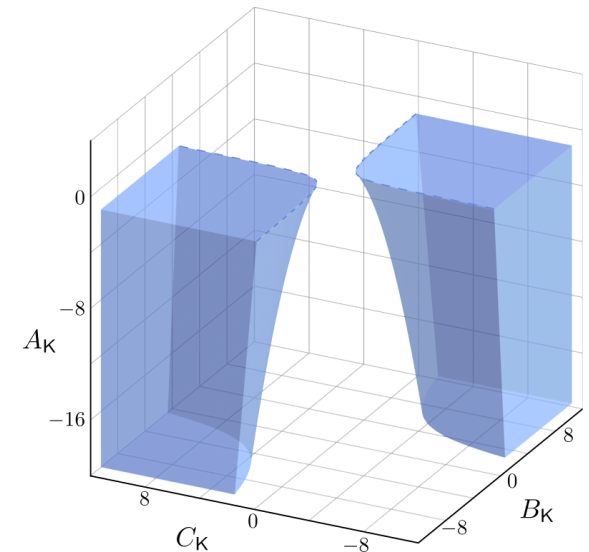
$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

Disconnected feasible region

- **Routh--Hurwitz stability criterion**

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_\mathsf{K} \\ B_\mathsf{K} & A_\mathsf{K} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, \begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix} \text{ is stable} \right\}$$

$$= \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_\mathsf{K} \\ B_\mathsf{K} & A_\mathsf{K} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_\mathsf{K} < -1, \ B_\mathsf{K}C_\mathsf{K} < A_\mathsf{K} \right\}.$$

- **Two path-connected components**

$$\mathcal{C}_1^+ := \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_\mathsf{K} \\ B_\mathsf{K} & A_\mathsf{K} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_\mathsf{K} < -1, \ B_\mathsf{K}C_\mathsf{K} < A_\mathsf{K}, \ B_\mathsf{K} > 0 \right\},$$

$$\mathcal{C}_1^- := \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_\mathsf{K} \\ B_\mathsf{K} & A_\mathsf{K} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_\mathsf{K} < -1, \ B_\mathsf{K}C_\mathsf{K} < A_\mathsf{K}, \ B_\mathsf{K} < 0 \right\}.$$

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Proof idea: Lifting via Change of Variables

☐ **Change of variables in state-space domain: Lyapunov theory**
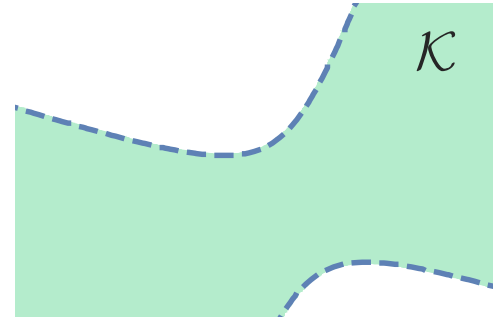
- Connectivity of the static stabilizing state feedback gains

$$\{K \in \mathbb{R}^{m \times n} \mid A - BK \text{ is stable}\}$$

$$\Longleftrightarrow \{K \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, P(A - BK)^\mathsf{T} + (A - BK)P \prec 0\}$$

$$\Longleftrightarrow \{K \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, PA^\mathsf{T} - L^\mathsf{T}B^\mathsf{T} + AP - BL \prec 0, L = KP\}$$

$$\Longleftrightarrow \{K = LP^{-1} \in \mathbb{R}^{m \times n} \mid \exists P \succ 0, PA^\mathsf{T} - L^\mathsf{T}B^\mathsf{T} + AP - BL \prec 0\}.$$

$\mathcal{K}$

Open, connected,
possibly nonconvex

- How about the set of stabilizing dynamical controllers

$$\begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix} \text{ is stable}$$

Change of variables for output feedback control is highly non-trivial

$$\Longleftrightarrow \quad \exists P \succ 0, P \begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix}^\mathsf{T} + \begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix} P \prec 0,$$
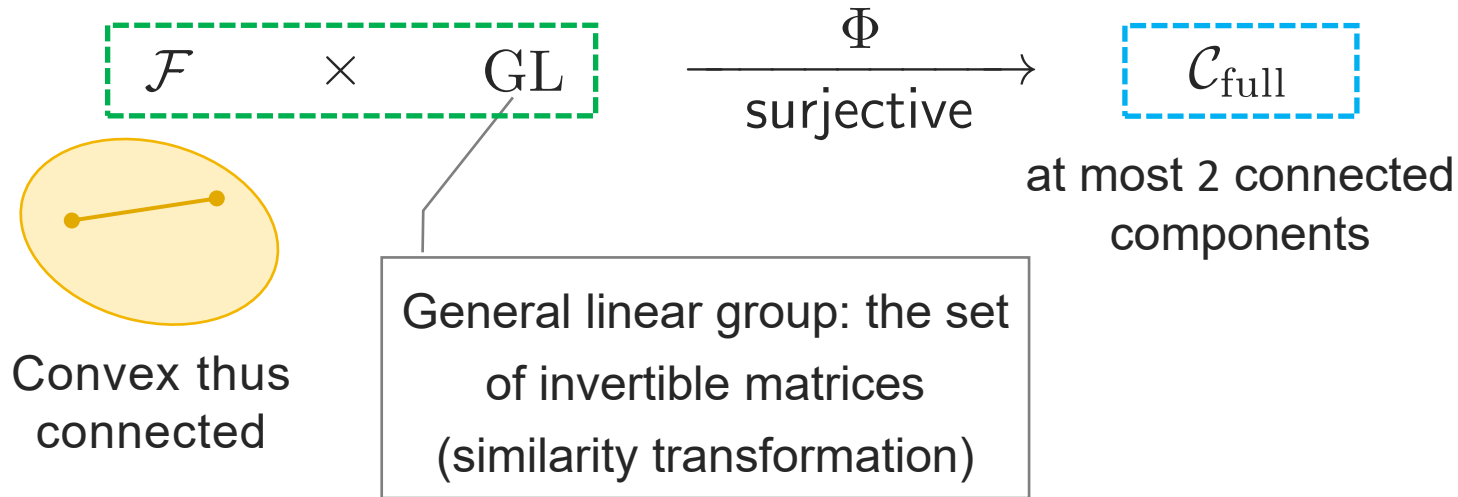
[Gahinet and Apkarian, 1994]
[Scherer et al., IEEE TAC 1997]

Proof idea: Lifting via Change of Variables

❑ **Change of variables in state-space domain: Lyapunov theory**

$$\Phi(\mathsf{Z}) = \begin{bmatrix} \Phi_D(\mathsf{Z}) & \Phi_C(\mathsf{Z}) \\ \Phi_B(\mathsf{Z}) & \Phi_A(\mathsf{Z}) \end{bmatrix} := \begin{bmatrix} I & 0 \\ YB & \Xi \end{bmatrix}^{-1} \begin{bmatrix} G & H \\ F & M-YAX \end{bmatrix} \begin{bmatrix} I & CX \\ 0 & \Pi \end{bmatrix}^{-1}.$$

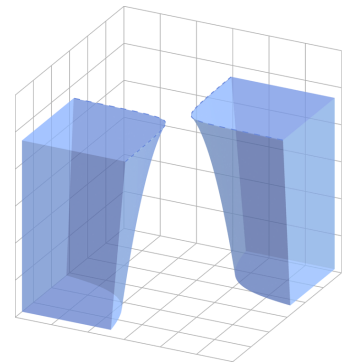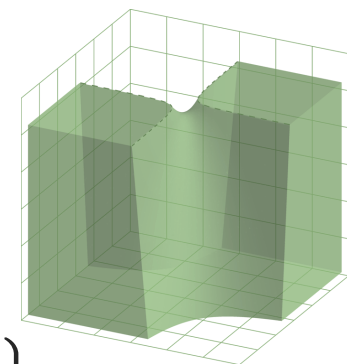[Scherer et al., IEEE TAC 1997]
[Gahinet and Apkarian, 1994]

$$\boxed{\mathcal{F} \qquad \times \qquad \mathrm{GL}} \xrightarrow[\text{surjective}]{\Phi} \boxed{\mathcal{C}_{\text{full}}}$$

at most 2 connected
components

Two connected components

$$\mathrm{GL}_n^+ = \{\Pi \in \mathbb{R}^{n\times n} \mid \det \Pi > 0\},$$
$$\mathrm{GL}_n^- = \{\Pi \in \mathbb{R}^{n\times n} \mid \det \Pi < 0\}.$$

Convex thus
connected

General linear group: the set
of invertible matrices
(similarity transformation)

$$\mathcal{F} = \Big\{ (X,Y,M,H,F) \mid X,Y \in \mathbb{S}^n, M \in \mathbb{R}^{n\times n}, H \in \mathbb{R}^{n\times p}, F \in \mathbb{R}^{m\times n},$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0, \begin{bmatrix} AX+BF & A \\ M & YA+HC \end{bmatrix} + \begin{bmatrix} AX+BF & A \\ M & YA+HC \end{bmatrix}^\top \prec 0 \Big\}$$

Model-free Optimization formulation
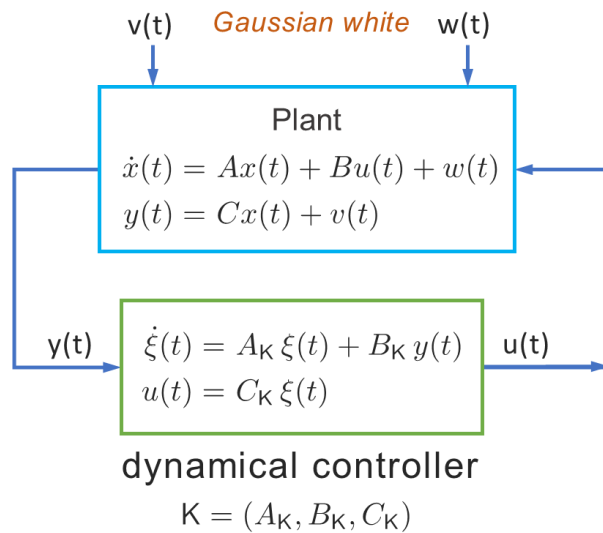
v(t)   *Gaussian white*   w(t)

Plant
$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = Cx(t) + v(t)$$

y(t)
$$\dot{\xi}(t) = A_{\mathsf{K}}\,\xi(t) + B_{\mathsf{K}}\,y(t)$$
$$u(t) = C_{\mathsf{K}}\,\xi(t)$$
u(t)

dynamical controller
$$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$$

Landscape Analysis

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$

$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

- **Q1: Connectivity of the feasible region** $\mathcal{C}_{\text{full}}$

  - Is it connected? **No**

  - How many connected components can it have? **Two**

- **Q2: Structure of stationary points of** $J(\mathsf{K})$

  - Are there spurious (strictly suboptimal, saddle) stationary points?

  - How to check if a stationary point is globally optimal?

Outline

Structure of Stationary Points

❑ Simple observations

1) $J(\mathsf{K})$ is a real analytic function over its domain (smooth, infinitely differentiable)
2) $J(\mathsf{K})$ has **non-unique** and **non-isolated** global optima

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$

$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

$$\dot{\xi}(t) = A_{\mathsf{K}}\,\xi(t) + B_{\mathsf{K}}\,y(t)$$
$$u(t) = C_{\mathsf{K}}\,\xi(t)$$

Similarity transformation

$$(A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \mapsto (T A_{\mathsf{K}} T^{-1}, T B_{\mathsf{K}}, C_{\mathsf{K}} T^{-1})$$

➤ $J(\mathsf{K})$ is invariant under similarity transformations.

➤ It has many stationary points, unlike the LQR with a unique stationary point
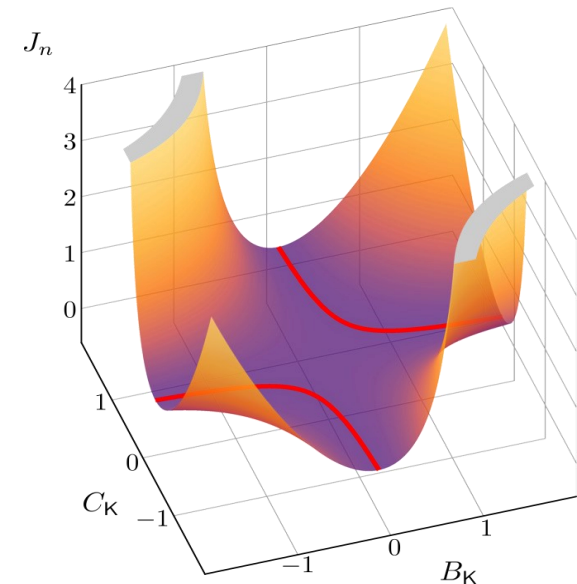
Structure of Stationary Points

❑ Gradient computation

Lemma 1: For every $\mathsf{K} = (A_\mathsf{K}, B_\mathsf{K}, C_\mathsf{K}) \in \mathcal{C}_{\text{full}}$, we have

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 2\left(Y_{12}^\mathsf{T} X_{12} + Y_{22} X_{22}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 2\left(Y_{22} B_\mathsf{K} V + Y_{22} X_{12}^\mathsf{T} C^\mathsf{T} + Y_{12}^\mathsf{T} X_{11} C^\mathsf{T}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 2\left(R C_\mathsf{K} X_{22} + B^\mathsf{T} Y_{11} X_{12} + B^\mathsf{T} Y_{12} X_{22}\right),$$

where $X_\mathsf{K} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^\mathsf{T} & X_{22} \end{bmatrix}$, $Y_\mathsf{K} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^\mathsf{T} & Y_{22} \end{bmatrix}$
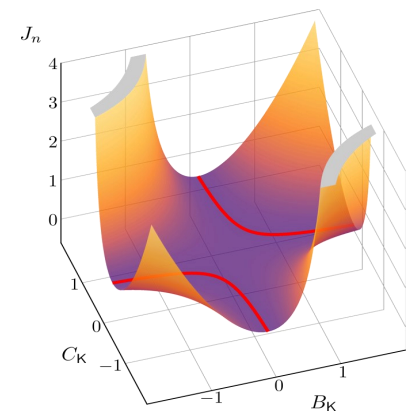
are the unique positive semidefinite solutions to two Lyapunov equations.

How does the set of Stationary Points look like?

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \ \middle| \ \begin{array}{l} \dfrac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0, \\[2mm] \dfrac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 0, \\[2mm] \dfrac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 0, \end{array} \right\}$$

❑ Non-unique, non-isolated

❑ Local minimum, local maximum, saddle points, or globally minimum?

Structure of Stationary Points

❑ **Main Result: existences of strict saddle points**

Theorem 4: Consider any open-loop stable plant. The zero controller with any stable A_{K}

$$\mathsf{K} = (A_{\mathsf{K}}, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero.

Example: $\dot{x}(t) = -x(t) + u(t) + w(t)$ $Q = 1, R = 1, V = 1, W = 1$

$y(t) = x(t) + v(t)$

Stationary point: $\mathsf{K}^{\star} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \in \mathbb{R}^{2\times 2}$,   with $a < 0$

➤ **Cost function:** $J\left(\begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix}\right) = \dfrac{A_{\mathsf{K}}^2 - A_{\mathsf{K}}(1 + B_{\mathsf{K}}^2 C_{\mathsf{K}}^2) - B_{\mathsf{K}}C_{\mathsf{K}}(1 - 3B_{\mathsf{K}}C_{\mathsf{K}} + B_{\mathsf{K}}^2 C_{\mathsf{K}}^2)}{2(-1 + A_{\mathsf{K}})(A_{\mathsf{K}} + B_{\mathsf{K}}C_{\mathsf{K}})}.$

➤ **Hessian:** $\begin{bmatrix} \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}}^2} & \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}}\partial B_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}}\partial C_{\mathsf{K}}} \\ \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}} A_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}}^2} & \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}}\partial C_{\mathsf{K}}} \\ \frac{\partial J^2(\mathsf{K})}{\partial C_{\mathsf{K}} A_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial C_{\mathsf{K}} B_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial\partial C_{\mathsf{K}}^2} \end{bmatrix}\Bigg|_{\mathsf{K}^{\star} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}} = \dfrac{1}{2(1-a)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$

Indefinite with eigenvalues:

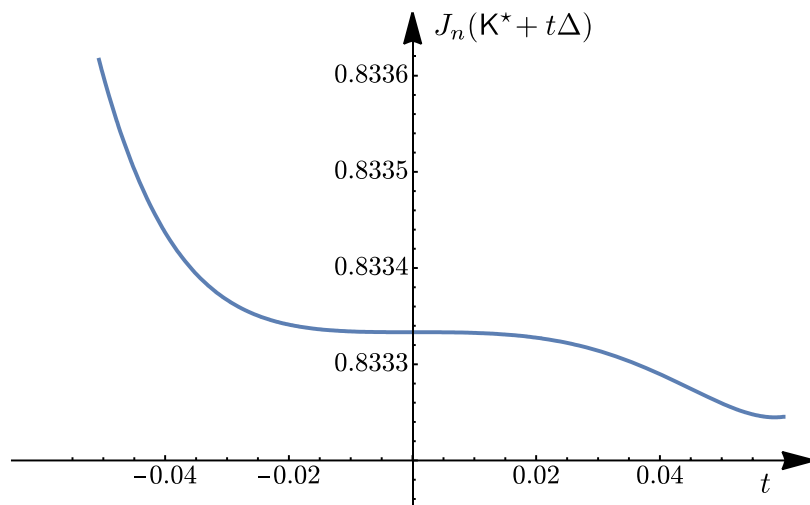0 and $\pm\frac{1}{2(1-a)}$

Structure of Stationary Points

❑ **Main Result: existences of strict saddle points**

Theorem 4: Consider any open-loop stable plant. The zero controller with any stable A_K

$$\mathsf{K} = (A_K, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero.

Another example with zero Hessian

How does the set of Stationary Points look like?

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \, \middle| \, \begin{array}{l} \dfrac{\partial J(\mathsf{K})}{\partial A_K} = 0, \\[6pt] \dfrac{\partial J(\mathsf{K})}{\partial B_K} = 0, \\[6pt] \dfrac{\partial J(\mathsf{K})}{\partial C_K} = 0, \end{array} \right\}$$

❑ **Non-unique, non-isolated**

❑ **Strictly suboptimal points; Strict saddle points**

❑ **All bad stationary points correspond to non-minimal controllers**

Structure of Stationary Points

❑ Main Result

Theorem 5:

> All stationary points corresponding to controllable and observable controllers are globally optimum.

Particularly, given a stationary point that is a **minimal** controller

1) This stationary point is a global optimum of $\mathcal{J}(\mathsf{K})$

2) The set of all global optima forms a manifold with 2 connected components. They are connected by a similarity transformation.

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \,\middle|\, \begin{array}{l} \dfrac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 0, \\[2mm] \dfrac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 0, \\[2mm] \dfrac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 0, \end{array} \right\}$$

Example: open-loop unstable system

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

Example: open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

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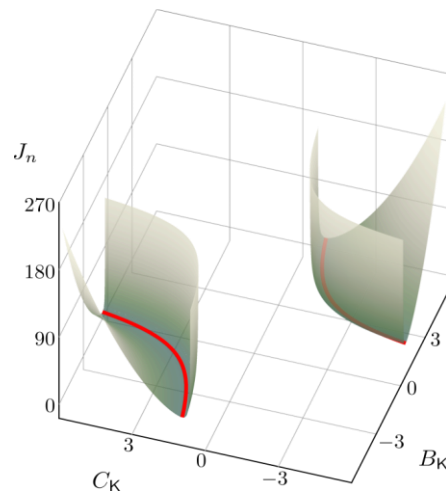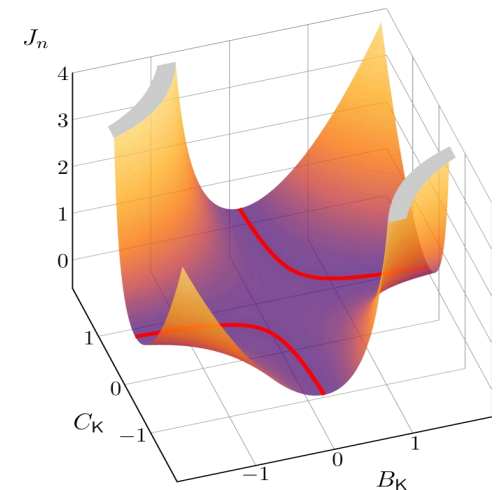
Proof idea

❑ **Proof: all minimal stationary points are unique up to a similarity transformation**

All minimal stationary points $\mathsf{K} = (A_\mathsf{K}, B_\mathsf{K}, C_\mathsf{K}) \in \mathcal{C}_{\mathrm{full}}$ to the LQG problem are in the form of

$$A_\mathsf{K} = T(A - BK - LC)T^{-1}, \qquad B_\mathsf{K} = -TL, \qquad C_\mathsf{K} = KT^{-1},$$

$$K = R^{-1}B^\mathsf{T}S, \ L = PC^\mathsf{T}V^{-1},$$

T is an invertible matrix and P, S are the unique positive definite solutions to the Riccati equations

$$\begin{cases} & \left. \dfrac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0, \right. \\ \mathsf{K} \in \mathcal{C}_{\mathrm{full}} & \left| \ \dfrac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 0, \right. \\ & \left. \dfrac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 0, \right. \end{cases}$$

Minimal controller

$$\frac{\partial J_n(\mathsf{K})}{\partial B_\mathsf{K}} = 0 \qquad \Longrightarrow \qquad B_\mathsf{K} = -TPC^\mathsf{T}V^{-1}$$

$$\frac{\partial J_n(\mathsf{K})}{\partial C_\mathsf{K}} = 0 \qquad \Longrightarrow \qquad C_\mathsf{K} = R^{-1}B^\mathsf{T}ST^{-1}$$

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0 \qquad \Longrightarrow \qquad A_\mathsf{K} = T(A - PC^\mathsf{T}V^{-1}C - BR^{-1}B^\mathsf{T}S)T^{-1}$$

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 2\left(Y_{12}^\mathsf{T}X_{12} + Y_{22}X_{22} \right),$$

$$\frac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 2\left(Y_{22}B_\mathsf{K}V + Y_{22}X_{12}^\mathsf{T}C^\mathsf{T} + Y_{12}^\mathsf{T}X_{11}C^\mathsf{T} \right),$$

$$\frac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 2\left(RC_\mathsf{K}X_{22} + B^\mathsf{T}Y_{11}X_{12} + B^\mathsf{T}Y_{12}X_{22} \right),$$

Special case in Theorem 20.6 of Zhou et al., 1996 and Section II of Hyland, 1984

Structure of Stationary Points

❏ **Implication**

Corollary: Consider gradient descent iterations

$$\mathsf{K}_{t+1} = \mathsf{K}_t - \alpha \nabla J(\mathsf{K}_t)$$

If the iterates converge to a minimal controller, then this minimal controller is a global optima.
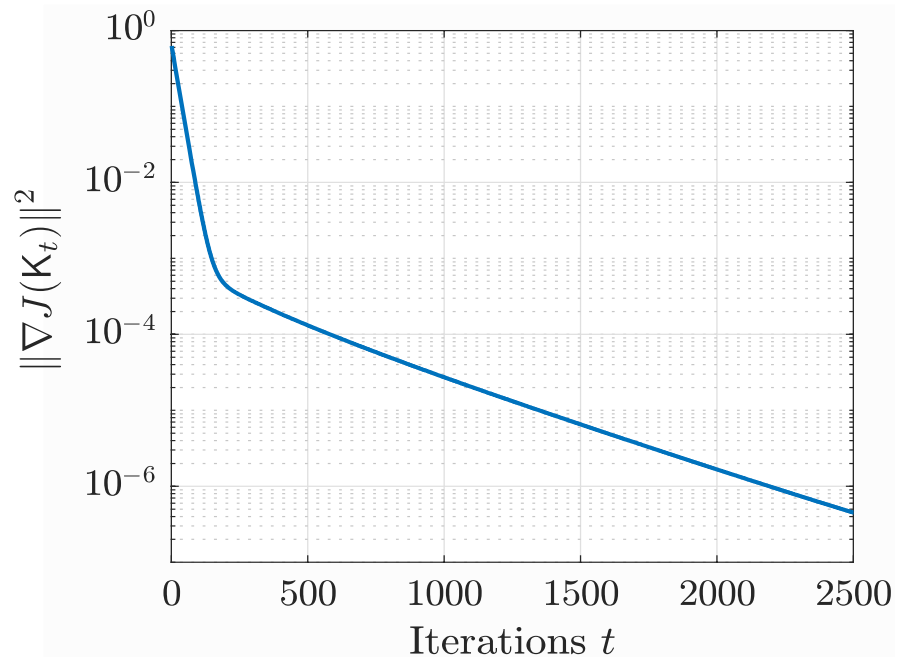
Open questions:

✓ Convergence conditions?

✓ Convergence speed?

✓ Alternative model-free parameterization

A very recent and related paper on output estimation is

Umenberger, J., Simchowitz, M., Perdomo, J. C., Zhang, K., & Tedrake, R. (2022). Globally Convergent Policy Search over Dynamic Filters for Output Estimation. *arXiv preprint arXiv:2202.11659*.

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Comparison with LQR

|	LQR as an Optimization problem $$\min_{K} \quad J(K)$$ $$\text{s.t.} \quad K \in \mathcal{K}$$	LQG as an Optimization problem $$\min_{\mathsf{K}} \quad J(\mathsf{K})$$ $$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$
Connectivity of feasible region	❖ Always connected	❖ Disconnected, but at most 2 connected comp. ❖ They are almost identical to each other
Stationary points	❖ Unique	❖ Non-unique, non-isolated stationary points ❖ Spurious stationary points (strict saddle, nonminimal controller) ❖ **All mini. stationary points are globally optimal**
Gradient Descent	❖ Gradient dominance ❖ Global fast convergence (like strictly convex)	❖ No gradient dominance ❖ Local convergence/speed (**unknown**) ❖ **Many open questions**
References	Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others	Zheng*, Tang*, Li. 2021, link (* equal contribution)

Outline

Landscape Analysis of LQG control

❑ Much richer and more complicated than LQR

❑ Disconnected, but at most 2 connected components

❑ Non-unique, non-isolated stationary points, strict saddle points

❑ Minimal stationary points are globally optimal
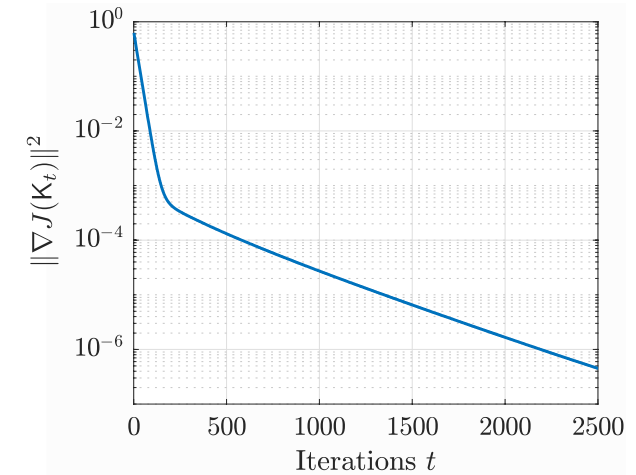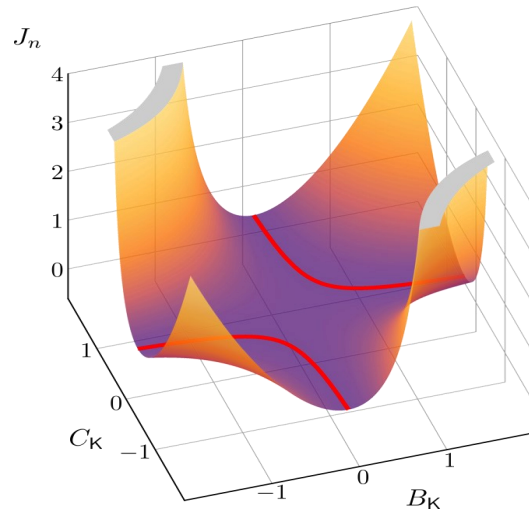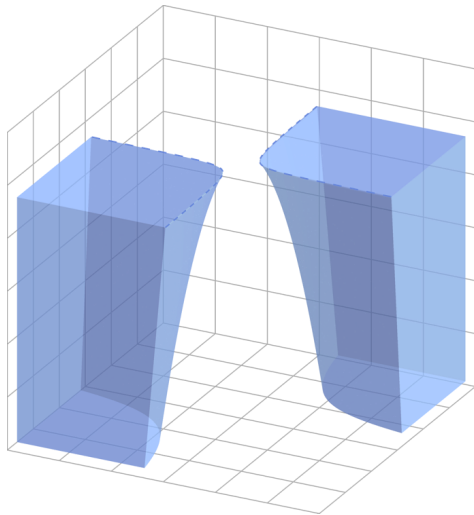
Ongoing and Future work

❑ A comprehensive classification of stationary points

❑ Conditions for existence of minimal globally optimal controllers

❑ Saddle points with vanishing Hessians may exist. How to deal with them?

❑ Alternative model-free parametrization of dynamic controllers

  ✓ Better optimization landscape structures, smaller dimension

❑ Perturbed policy gradient?

Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

Thank you for your attention!

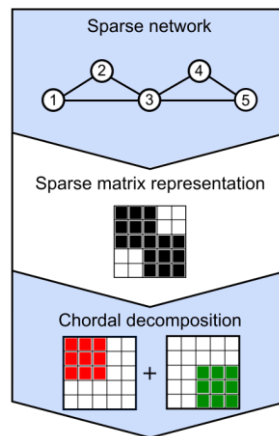Q & A

More details. Check out our paper: https://arxiv.org/abs/2102.04393

SOC lab at UC San Diego
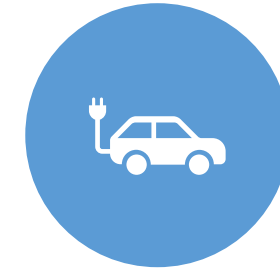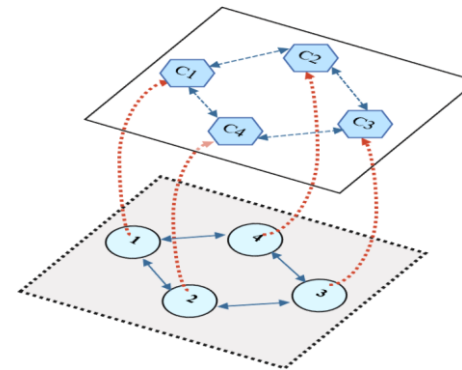
SOC Lab

Data-driven and learning-based control
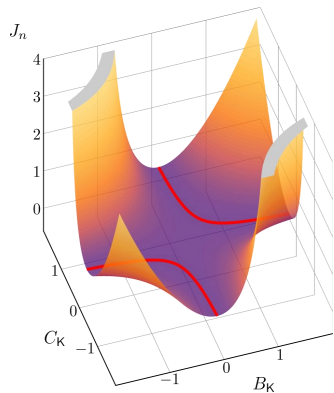
Sparse conic optimization
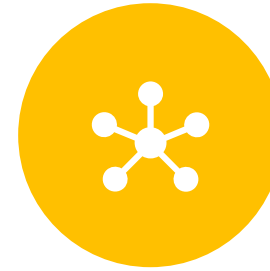
Scalable distributed control

Connected and autonomous vehicles (CAVs)