Scalable Semidefinite and Sum-of-square Optimization via Matrix Decomposition

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Acknowledgments

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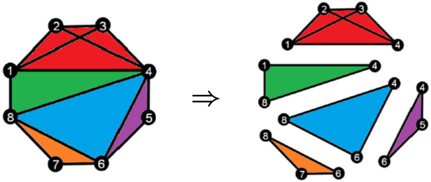
Outline

1. Introduction: Matrix decomposition and chordal graphs
2. Part I - Decomposition in sparse semidefinite optimization
3. Part II - Decomposition in sparse sum-of-squares optimization
4. Part III - Beyond chordal decomposition
5. Conclusion

Introduction: Matrix decomposition and chordal graphs



Matrix decomposition and chordal graphs

Matrix decomposition:

- A simple example

$$A = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{\succeq 0}$$

- This is true for any PSD matrix with such pattern, *i.e.*, sparse cone decomposition

$$\underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}$$

  where $*$ denotes a real scalar number.

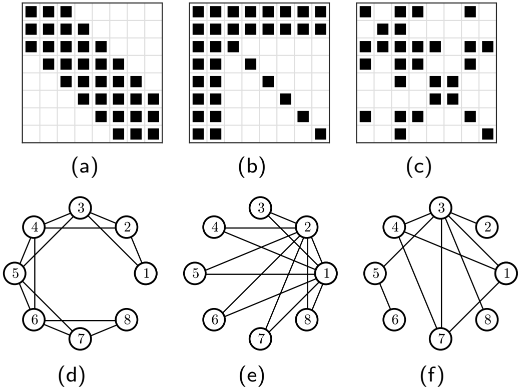
Benefits:

- Reduce computational complexity, and thus improve efficiency! $(3 \times 3 \rightarrow 2 \times 2)$

Matrix decomposition and chordal graphs

Matrix decomposition:

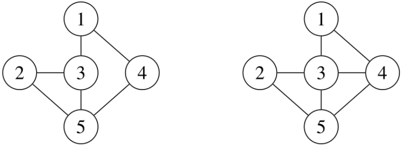
- Many other patterns admit similar decompositions, *e.g.*



- They can be commonly characterized by **chordal graphs**.

Matrix decomposition and chordal graphs

Chordal graphs: An undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *chordal* if every cycle of length greater than three has a chord.

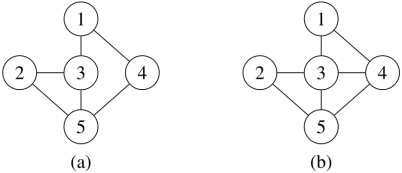


(a) (b)

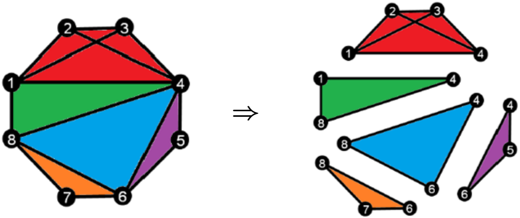
Notation: (Vandenberghe, & Andersen, 2014)

- *Chordal extension:* Any non-chordal graph can be chordal extended;
- *Maximal clique:* A clique is a set of nodes that induces a complete subgraph;
- *Clique decomposition:* A chordal graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ can be decomposed into a set of maximal cliques $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$.

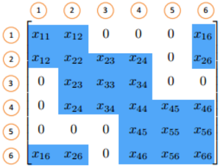
Chordal graphs: An undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is called *chordal* if every cycle of length greater than three has a chord.

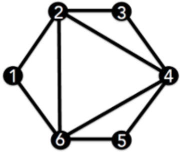


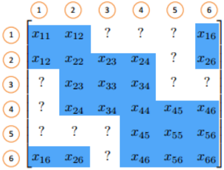
Clique decomposition:



Matrix decomposition and chordal graphs







Sparse positive semidefinite (PSD) matrices

$$\mathbb{S}^n(\mathcal{E},0) = \{X \in \mathbb{S}^n \mid X_{ij} = X_{ji} = 0, \forall (i,j) \notin \mathcal{E}\},$$
$$\mathbb{S}^n_+(\mathcal{E},0) = \{X \in \mathbb{S}^n(\mathcal{E},0) \mid X \succeq 0\}.$$

Positive semidefinite completable matrices

$$\mathbb{S}^n(\mathcal{E},?) = \{X \in \mathbb{S}^n \mid X_{ij} = X_{ji}, \text{ given if } (i,j) \in \mathcal{E}\},$$
$$\mathbb{S}^n_+(\mathcal{E},?) = \{X \in \mathbb{S}^n(\mathcal{E},?) \mid \exists M \succeq 0, M_{ij} = X_{ij}, \forall (i,j) \in \mathcal{E}\}.$$

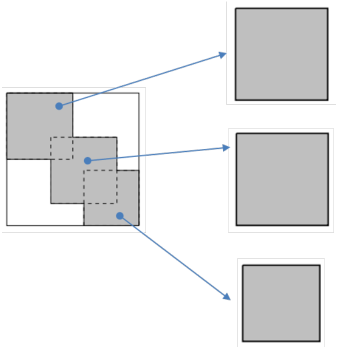
$\mathbb{S}^n_+(\mathcal{E},0)$ and $\mathbb{S}^n_+(\mathcal{E},?)$ are dual to each other.

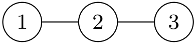
Matrix decomposition and chordal graphs

Clique decomposition for PSD completable matrices (Grone, *et al.*, 1984)

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a chordal graph with maximal cliques $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_p\}$. Then,

$$X \in \mathbb{S}_+^n(\mathcal{E}, ?) \Leftrightarrow E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^T \in \mathbb{S}_+^{|\mathcal{C}_k|}, \qquad k = 1, \ldots, p.$$





$$\begin{bmatrix} X_{11} & X_{12} & ? \\ X_{21} & X_{22} & X_{23} \\ ? & X_{32} & X_{33} \end{bmatrix} \in \mathbb{S}_+^3(\mathcal{E}, ?)$$

$$\Updownarrow$$

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0$$

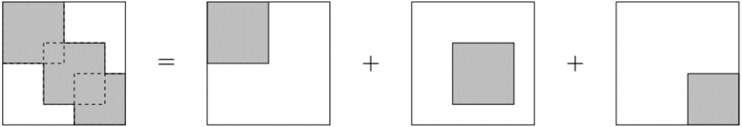
$$\begin{bmatrix} X_{22} & X_{23} \\ X_{32} & X_{33} \end{bmatrix} \succeq 0$$

Matrix decomposition and chordal graphs

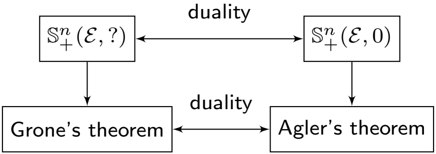
Clique decomposition for PSD matrices (Agler, *et al.*, 1988; Griewank and Toint, 1984)

Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be a chordal graph with maximal cliques $\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_p\}$. Then,

$$Z \in \mathbb{S}^n_+(\mathcal{E},0) \Leftrightarrow Z = \sum_{k=1}^{p} E_{\mathcal{C}_k}^T Z_k E_{\mathcal{C}_k},\ Z_k \in \mathbb{S}_+^{|\mathcal{C}_k|}$$



Sparse Cone Decomposition



Matrix decomposition and chordal graphs

Applications (a partial and incomplete list)

- **Sparse semidefinite programs** → Part I of the talk
  - Fukuda, Kojima, Murota, Nakata, 2001; Andersen, Dahl, Vandenberghe, 2010; Sun, Andersen, Vandenberghe, 2014; Madani, Kalbat, Lavaei, 2015; Zheng, Fantuzzi, Papachristodoulou, Goulart, Wynn, 2017;
- Analysis and control of sparse networked systems
  - Andersen, Pakazad, Hansson, Rantzer, 2014; Mason, Papachristodoulou, 2014; Zheng, Mason, Papachristodoulou, 2018; Pakazad, Hansson, Andersen, Rantzer, 2018; Zheng, Kamgarpour, Sootla, Papachristodoulou, 2018.
- Power systems (OPF problems)
  - Dall'Anese, Zhu, Giannakis, 2013; Andersen, Hansson, Vandenberghe, 2014
- **Polynomial optimization** → Part II of the talk
  - Waki, Kim, Kojima, Muramatsu, 2006; Lasserre, 2006; Fawzi, Saunderson, Parrilo, 2016.

A survey paper

- Vandenberghe, Lieven, and Martin S. Andersen. "Chordal graphs and semidefinite optimization." Foundations and Trends in Optimization 1.4 (2015): 241-433.

Part I: Decomposition in sparse semidefinite optimization

Sparse semidefinite programs (SDPs)

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{subject to} \quad & \langle A_i, X \rangle = b_i, i = 1, \ldots, m, \\ & X \succeq 0. \end{aligned}$$

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y \rangle \\ \text{subject to} \quad & Z + \sum_{i=1}^{m} A_i\, y_i = C, \\ & Z \succeq 0. \end{aligned}$$

where $X \succeq 0$ means X is positive semidefinite.

- **Applications:** Control theory, fluid dynamics, polynomial optimization, *etc*.
- **Interior-point solvers:** SeDuMi, SDPA, SDPT3 (suitable for small and medium-sized problems); *Modelling package:* YALMIP, CVX
- **Large-scale cases:** it is important to exploit the inherent structure
  - Low rank;
  - Algebraic symmetry;
  - **Chordal sparsity**
    - Second-order methods: Fukuda *et al.*, 2001; Nakata *et al.*, 2003; Burer 2003; Andersen *et al.*, 2010.
    - **First-order methods**: Madani *et al.*, 2015; Sun, Andersen, and Vandenberghe, 2014.

Aggregate sparsity pattern of matrices

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}$$

Primal SDP

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{subject to} \quad & \langle A_1, X \rangle = b_1 \\ & \langle A_2, X \rangle = b_2 \\ & X \succeq 0. \end{aligned}$$

Dual SDP

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y \rangle \\ \text{subject to} \quad & y_1 A_1 + y_2 A_2 + Z = C, \\ & Z \succeq 0. \end{aligned}$$

$X \in \begin{bmatrix} * & * & ? \\ * & * & * \\ ? & * & * \end{bmatrix}$	Patterns of feasible solutions	$Z \in \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}$
$X \in \mathbb{S}^3_+(\mathcal{E}, ?)$	Cone replacement	$Z \in \mathbb{S}^3_+(\mathcal{E}, 0)$

Apply the clique decomposition on $\mathbb{S}^3_+(\mathcal{E}, ?)$ and $\mathbb{S}^3_+(\mathcal{E}, 0)$

- Fukuda *et al.*, 2001; Nakata *et al.*, 2003; Andersen *et al.*, 2010; Madani *et al.*, 2015; Sun, Andersen, and Vandenberghe, 2014.

Cone decomposition of sparse SDPs

Primal SDP

$$\begin{aligned} \min \quad & \langle C, X\rangle \\ \text{subject to} \quad & \langle A_i, X\rangle = b_i, i = 1, \ldots, m \\ & \boxed{X \succeq 0}. \end{aligned}$$

Dual SDP

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y\rangle \\ \text{subject to} \quad & \sum_{i=1}^{m} y_i A_i + Z = C, \\ & \boxed{Z \succeq 0}. \end{aligned}$$

$X \in \mathbb{S}^n_+(\mathcal{E}, ?)$

Cone replacement

(Assuming an aggregate sparsity pattern $\mathcal{E}$)

$Z \in \mathbb{S}^n_+(\mathcal{E}, 0)$

$\Downarrow$ $\Downarrow$

$$\begin{aligned} \min \quad & \langle C, X\rangle \\ \text{s.t.} \quad & \langle A_i, X\rangle = b_i, i = 1, \ldots, m \\ & \boxed{E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^T \succeq 0, k = 1, \ldots, p}. \end{aligned}$$

$$\begin{aligned} \max_{y,\, Z} \quad & \langle b, y\rangle \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i A_i + \sum_{k=1}^{p} E_{\mathcal{C}_k}^T Z_k E_{\mathcal{C}_k} = C, \\ & \boxed{Z_k \succeq 0, k = 1, \ldots, p} \end{aligned}$$

- **A big sparse PSD cone** is equivalently replaced by a set of **coupled small** PSD cones;
- Our idea: **consensus** variables $\Rightarrow$ decouple the coupling constraints;

Decomposed SDPs for operator-splitting algorithms

Primal decomposed SDP

$$\begin{aligned} \min_{X, X_k} \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \qquad i = 1, \ldots, m, \\ & \boxed{X_k = E_{\mathcal{C}_k} X E_{\mathcal{C}_k}^T, k = 1, \ldots, p,} \\ & X_k \in \mathbb{S}_+^{|\mathcal{C}_k|}, \qquad k = 1, \ldots, p. \end{aligned}$$

Dual decomposed SDP

$$\begin{aligned} \max_{y, Z_k, V_k} \quad & \langle b, y \rangle \\ \text{s.t.} \quad & \sum_{i=1}^{m} A_i \, y_i + \sum_{k=1}^{p} E_{\mathcal{C}_k}^T V_k E_{\mathcal{C}_k} = C, \\ & \boxed{Z_k - V_k = 0, \ k = 1, \ldots, p,} \\ & Z_k \in \mathbb{S}_+^{|\mathcal{C}_k|}, \quad k = 1, \ldots, p. \end{aligned}$$

- A set of slack consensus variables has been introduced;
- The slack variables allow one to **separate the conic and the affine constraints** when using operator-splitting algorithms $\Rightarrow$ fast iterations

Vectorization

$$\begin{aligned} \min_{x, x_k} \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax = b, \\ & \boxed{x_k = H_k x}, \quad k = 1, \ldots, p, \\ & x_k \in \mathcal{S}_k, \qquad k = 1, \ldots, p, \end{aligned}$$

$$\begin{aligned} \max_{y, z_k, v_k} \quad & \langle b, y \rangle \\ \text{s.t.} \quad & A^T y + \sum_{k=1}^{p} H_k^T v_k = c, \\ & \boxed{z_k - v_k = 0}, k = 1, \ldots, p, \\ & z_k \in \mathcal{S}_k, k = 1, \ldots, p. \end{aligned}$$

Alternating Direction Method of Multipliers (ADMM)

The ADMM algorithm solves the optimization problem (Bertsekas and Tsitsiklis, 1989; Boyd, *et al.*, 2011)

$$\begin{aligned} \min_{x,y} \quad & f(x) + g(y) \\ \text{subject to} \quad & Ax + By = c, \end{aligned}$$

where f and g are convex functions.

- **Augmented Lagrangian**

$$\mathcal{L}_\rho(x, y, z) := f(x) + g(y) + z^T \left(Ax + By - c\right) + \frac{\rho}{2} \left\|Ax + By - c\right\|^2$$

- **ADMM steps**

$$\begin{aligned} x^{(n+1)} &= \arg\min_x \; \mathcal{L}_\rho(x, y^{(n)}, z^{(n)}), && \rightarrow x\text{-minimization step} \\ y^{(n+1)} &= \arg\min_y \; \mathcal{L}_\rho(x^{(n+1)}, y, z^{(n)}), && \rightarrow y\text{-minimization step} \\ z^{(n+1)} &= z^{(n)} + \rho \left(Ax^{(n+1)} + By^{(n+1)} - c\right). && \rightarrow \text{dual variable update} \end{aligned}$$

ADMM is particularly suitable when the subproblems have closed-form expressions, or can be solved efficiently.

ADMM for primal decomposed SDPs

$$\begin{aligned} \min_{x, x_k} \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax = b, \\ & \boxed{x_k = H_k x}, \quad k = 1, \ldots, p, \\ & x_k \in \mathcal{S}_k, \qquad k = 1, \ldots, p, \end{aligned}$$

Reformulation using indicator functions

$$\begin{aligned} \min_{x, x_1, \ldots, x_p} \quad & \langle c, x \rangle + \delta_0 \left(Ax - b\right) + \sum_{k=1}^{p} \delta_{\mathcal{S}_k}(x_k) \\ \text{s.t.} \quad & x_k = H_k x, \quad k = 1, \ldots, p. \end{aligned}$$

- *x-minimization step:* QP with linear constraints, KKT condition

$$\begin{bmatrix} D & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{p} H_k^T \left(x_k^{(n)} + \rho^{-1} \lambda_k^{(n)} \right) - \rho^{-1} c \\ b \end{bmatrix}.$$

- *y-minimization step:* Parallel projections onto **small PSD cones**

$$\begin{aligned} \min_{x_k} \quad & \left\| x_k - H_k x^{(n+1)} + \rho^{-1} \lambda_k^{(n)} \right\|^2 \\ \text{s.t.} \quad & x_k \in \mathcal{S}_k. \end{aligned}$$

- Update multipliers

ADMM for dual decomposed SDPs

$$\begin{aligned}\max_{y, z_k, v_k} \quad & \langle b, y \rangle \\ \text{s.t.} \quad & A^T y + \sum_{k=1}^{p} H_k^T v_k = c, \\ & \boxed{z_k - v_k = 0}, k = 1, \ldots, p, \\ & z_k \in \mathcal{S}_k, k = 1, \ldots, p.\end{aligned}$$

Reformulation using indicator functions

$$\begin{aligned}\min \quad & -\langle b, y \rangle + \delta_0 \left(c - A^T y - \sum_{k=1}^{p} H_k^T v_k \right) + \sum_{k=1}^{p} \delta_{\mathcal{S}_k}(z_k) \\ \text{s.t.} \quad & z_k = v_k, \quad k = 1, \ldots, p.\end{aligned}$$

- *x-minimization step:* QP with linear constraints, KKT condition

$$\begin{bmatrix} D & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c - \sum_{k=1}^{p} H_k^T \left(z_k^{(n)} + \rho^{-1} \lambda_k^{(n)} \right) \\ -\rho^{-1} b \end{bmatrix},$$

- *y-minimization step:* Parallel projections onto **small PSD cones**

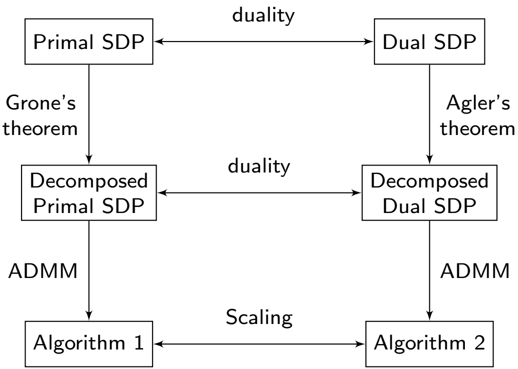
$$\begin{aligned}\min_{z_k} \quad & \left\| z_k - v_k^{(n)} + \rho^{-1} \lambda_k^{(n)} \right\|^2 \\ \text{s.t.} \quad & z_k \in \mathcal{S}_k.\end{aligned}$$

- Update multipliers

ADMM for primal and dual decomposed SDPs

Equivalence between the primal and dual cases

- ADMM steps in the dual form are scaled versions of those in the primal form.
- Extension to the homogeneous self-dual embedding exists.



Both algorithms only require conic projections onto small PSD cones. **Complexity depends on the largest maximal cliques, instead of the original dimension!**

CDCS

Cone decomposition conic solver

- An open source MATLAB solver for sparse conic programs;
- CDCS supports constraints on the following cones:
  - Free variables
  - non-negative orthant
  - second-order cone
  - the positive semidefinite cone.
- Input-output format is in accordance with SeDuMi; Interface via YALMIP.
- Syntax: `[x,y,z,info] = cdcs(At,b,c,K,opts);`

Download from `https://github.com/OxfordControl/CDCS`

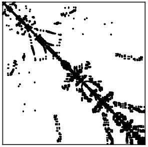
Numerical comparison

- SeDuMi (interior-point solver): default parameters, and low-accuracy solution 10^{-3}
- SCS (first-order solver)
- CDCS and SCS: stopping condition 10^{-3} (max. iterations 2000)
- All simulations were run on a PC with a 2.8 GHz Intel Core i7 CPU and 8GB of RAM.

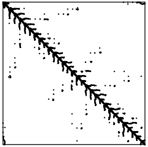
Large-scale sparse SDPs

Instances from Andersen, Dahl, Vandenberghe, 2010

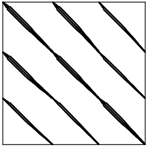
	rs35	rs200	rs228	rs365	rs1555	rs1907
Original cone size, n	2003	3025	1919	4704	7479	5357
Affine constraints, m	200	200	200	200	200	200
Number of cliques, p	588	1635	783	1244	6912	611
Maximum clique size	418	102	92	322	187	285
Minimum clique size	5	4	3	6	2	7



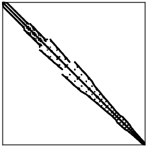
rs35



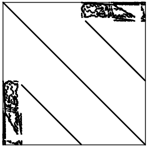
rs200



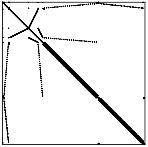
rs228



rs365



rs1555



rs1907

Large-scale sparse SDPs: Numerical results

	rs35			rs200		
	Time (s)	# Iter.	Objective	Time (s)	# Iter.	Objective
SeDuMi (high)	1 391	17	25.33	4 451	17	99.74
SeDuMi (low)	986	11	25.34	2 223	8	99.73
SCS (direct)	2 378	†2 000	25.08	9 697	†2 000	81.87
CDCS-primal	370	379	25.27	159	577	99.61
CDCS-dual	272	245	25.53	103	353	99.72
CDCS-hsde	208	198	25.64	54	214	99.77

	rs228			rs365		
	Time (s)	# Iter.	Objective	Time (s)	# Iter.	Objective
SeDuMi (high)	1 655	21	64.71	***	***	***
SeDuMi (low)	809	10	64.80	***	***	***
SCS (direct)	2 338	†2 000	62.06	34 497	†2 000	44.02
CDCS-primal	94	400	64.65	321	401	63.37
CDCS-dual	84	341	64.76	240	265	63.69
CDCS-hsde	38	165	65.02	151	175	63.75

	rs1555			rs1907		
	Time (s)	# Iter.	Objective	Time (s)	# Iter.	Objective
SeDuMi (high)	***	***	***	***	***	***
SeDuMi (low)	***	***	***	***	***	***
SCS (direct)	139 314	†2 000	34.20	50 047	†2 000	45.89
CDCS-primal	1 721	†2 000	61.22	330	349	62.87
CDCS-dual	317	317	69.54	271	252	63.30
CDCS-hsde	361	448	66.38	190	187	63.15

***: the problem could not be solved due to memory limitations.

†: maximum number of iterations reached.

Large-scale sparse SDPs: Numerical results

Average CPU time per iteration

	rs35	rs200	rs228	rs365	rs1555	rs1907
SCS (direct)	1.188	4.847	1.169	17.250	69.590	25.240
CDCS-primal	0.944	0.258	0.224	0.715	0.828	0.833
CDCS-dual	1.064	0.263	0.232	0.774	0.791	0.920
CDCS-hsde	1.005	0.222	0.212	0.733	0.665	0.891

- $20\times, 21\times, 26\times$, and $75\times$ faster than SCS, respectively, for problems rs200, rs365, rs1907, and rs1555.
- The computational benefit comes form the cone decomposition (projections onto small PSD cones)
- CDCS enables us to solve large, sparse conic problems with moderate accuracy that are beyond the reach of standard interior-point and/or other first-order methods

The conic projections in all Algorithms require $\mathcal{O}(\sum_{k=1}^{p} |\mathcal{C}_k|^3)$ flops. **Complexity is dominated by the largest maximal clique!**

Part II: Decomposition in sparse SOS optimization

— bridging the gap between DSOS/SDSOS optimization and SOS optimization

Checking nonnegativity and Sum-of-squares

Checking whether a given polynomial is nonnegative has applications in many areas.

$$p(x) = \sum p_\alpha x^\alpha \geq 0, \qquad \text{e.g.,} \quad p(x) = x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_2^2 \geq 0.$$

- **Application**: unconstrained polynomial optimization

$$\min_{x \in \mathbb{R}^n} p(x) \quad \Longleftrightarrow \quad \begin{aligned} \max \quad & \gamma \\ \text{subject to} \quad & p(x) - \gamma \geq 0. \end{aligned}$$

- **Sum-of-squares (SOS) relaxation**: $p(x)$ can be represented as a sum of finite squared polynomials $f_i(x), i = 1, \ldots, m$

$$p(x) = \sum_{i=1}^{m} f_i(x)^2,$$

- **SDP characterization (Parrilo 2000):** $p(x)$ is SOS if and only if there exists $Q \succeq 0$,

$$p(x) = v_d(x)^T Q v_d(x).$$

where $v_d(x) = [1, x_1, x_2, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d]^T$ is the standard monomial basis.

Checking nonnegativity and Sum-of-squares

Sum-of-square matrices

- Consider a symmetric matrix-valued polynomial

$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \dots & p_{1r}(x) \\ p_{21}(x) & p_{22}(x) & \dots & p_{2r}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{r1}(x) & p_{r2}(x) & \dots & p_{rr}(x) \end{bmatrix} \succeq 0, \forall x \in \mathbb{R}^n.$$

- Similar to the scalar case, the problem of checking whether $P(x)$ is positive semidefinite is NP-hard in general.
- **SOS relaxation:** We call $P(x)$ is an SOS matrix if

$$p(x,y) = y^T P(x) y \text{ is SOS in } [x;y]$$

- **SDP characterization (similar to the scalar case) (Parrilo et al.):** $P(x)$ is an SOS matrix if and only if there exists $Q \succeq 0$, such that

$$P(x) = (I_r \otimes v_d(x))^T Q (I_r \otimes v_d(x)).$$

where Q is called the Gram matrix.

SOS optimization

A general optimization problem:

- **Scalar version:** Consider the following real-valued SOS program

$$\begin{aligned} \min_{u} \quad & w^T u \\ \text{subject to} \quad & p_0(x) + \sum_{h=1}^{t} u_h p_h(x) \ \text{ is SOS}, \end{aligned} \tag{1}$$

  where $p_0(x), p_h(x), h = 1, \ldots, t$ are given polynomials.

- **Matrix version:** Consider the following matrix-valued SOS program

$$\begin{aligned} \min_{u} \quad & w^T u \\ \text{subject to} \quad & P_0(x) + \sum_{h=1}^{t} u_h P_h(x) \ \text{ is SOS}, \end{aligned} \tag{2}$$

  where $P_0(x), P_h(x), h = 1, \ldots, t$ are given symmetric polynomial matrices .

- Both (1) and (2) can be equivalently reformulated into SDPs;
- One fundamental problem is the poor scalability to large-scale instances, since

$$\binom{n+d}{d} = \mathcal{O}(n^d).$$

Scaled-diagonally dominant SOS (SDSOS) and DSOS

A new concept of (S)DSOS by Ahmadi and Majumdar, 2017

- *Diagonally dominant (dd) matrix:* a symmetric matrix $A = [a_{ij}]$ is dd if

$$a_{ii} \geq \sum_{j \neq i} |a_{ij}|, \forall i = 1, \ldots, n.$$

- *Scaled-diagonally dominant (sdd) matrix:* a symmetric matrix $A = [a_{ij}]$ is sdd if there exists a PSD diagonal matrix D, such that

$$DAD \text{ is dd.}$$

- *DSOS polynomials:* $p(x) = v_d(x)^T Q v_d(x)$, where the Gram matrix Q is dd.
- *SDSOS polynomials:* $p(x) = v_d(x)^T Q v_d(x)$, where the Gram matrix Q is sdd.

LP and SOCP-based optimization (Ahmadi and Majumdar, 2017)

- Optimization over dd matrices or DSOS polynomials is a linear program (LP).
- Optimization over sdd matrices or SDSOS polynomials is a second-order cone program (SOCP).

The gap between DSOS/SDSOS and SOS

A brief summary

- **SOS:** $p(x) = v_d(x)^T Q v_d(x)$, where the Gram matrix Q is PSD $\longrightarrow$ SDP
- **SDSOS:** $p(x) = v_d(x)^T Q v_d(x)$, where the Gram matrix Q is sdd $\longrightarrow$ SOCP
- **DSOS:** $p(x) = v_d(x)^T Q v_d(x)$, where the Gram matrix Q is dd $\longrightarrow$ LP

Another viewpoint

- **SDP** is an optimization problem involving PSD constraints of dimension $N \times N$
- **SOCP** is an optimization problem involving PSD constraints of dimension 2×2
- **LP** is an optimization problem involving PSD constraints of dimension 1×1

What is missing? How about problems that involve PSD constraints of dimension $k \times k$, where $1 \leq k \leq N$

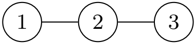
- One approach: factor-width k matrices (Boman, et al. 2005) $\longrightarrow$ Not practical $\binom{n}{k} = \mathcal{O}(n^k)$
- **Chordal decomposition**, considering sparsity and equivalent to sparse factor-width k matrices $\longrightarrow$ the main topic today.

Sparse polynomial matrix (similar to sparse real matrix)

- Given a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, we define a sparse polynomial matrix $P(x)$ where

$$p_{ij}(x) = 0, \text{ if } (i,j) \notin \mathcal{E}^*$$

- For example, for a line graph of three nodes



$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ & p_{32}(x) & p_{33}(x) \end{bmatrix}.$$

- Define a set of sparse polynomial matrices

$$\mathbb{R}^{r\times r}_{n,2d}(\mathcal{E},0) = \left\{ P(x) \in \mathbb{R}[x]^{r\times r}_{n,2d} \middle| p_{ij}(x) = p_{ji}(x) = 0, \text{ if } (i,j) \notin \mathcal{E}^* \right\}.$$

- SOS/SDSOS/DSOS matrices with a sparsity pattern $\mathcal{E}$

$$\begin{aligned} SOS^r_{n,2d}(\mathcal{E},0) &= SOS^r_{n,2d} \cap \mathbb{R}^{r\times r}_{n,2d}(\mathcal{E},0), \\ SDSOS^r_{n,2d}(\mathcal{E},0) &= SDSOS^r_{n,2d} \cap \mathbb{R}^{r\times r}_{n,2d}(\mathcal{E},0), \\ DSOS^r_{n,2d}(\mathcal{E},0) &= DSOS^r_{n,2d} \cap \mathbb{R}^{r\times r}_{n,2d}(\mathcal{E},0). \end{aligned}$$

Sparsity in SOS optimization

Sparsity in $P(x)$ does not necessarily lead to sparsity in the Gram matrix Q !!

For example

$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ & p_{32}(x) & p_{33}(x) \end{bmatrix} = \begin{bmatrix} v(x)^T Q_{11} v(x) & v(x)^T Q_{12} v(x) & v(x)^T Q_{13} v(x) \\ v(x)^T Q_{21} v(x) & v(x)^T Q_{22} v(x) & v(x)^T Q_{23} v(x) \\ v(x)^T Q_{31} v(x) & v(x)^T Q_{32} v(x) & v(x)^T Q_{33} v(x) \end{bmatrix}$$

$$= (I_3 \otimes v(x))^T \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} (I_3 \otimes v(x))$$

- If we make **a restriction that** $Q_{ij} = 0$**, if** $p_{ij}(x) = 0$, then the Gram matrix Q has the same pattern with $P(x)$. Now, **chordal decomposition** leads to

$$Q = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}$$

- We have the same chordal decomposition for polynomial matrix $P(x)$.

Sparse SOS matrix decomposition

Sparse version of SOS matrices

$$SSOS^r_{n,2d}(\mathcal{E},0) = \left\{ P(x) \in SOS^r_{n,2d}(\mathcal{E},0) \middle| P(x) \text{ admits a Gram matrix } Q \succeq 0, \text{ with } Q_{ij} = 0 \text{ when } p_{ij}(x) = 0 \right\}.$$

Theorem (Sparse SOS matrix decomposition)

If $\mathcal{E}$ is chordal with a set of maximal cliques $\mathcal{C}_1, \ldots, \mathcal{C}_t$, then

$$P(x) \in SSOS^r_{n,2d}(\mathcal{E},0) \Leftrightarrow P(x) = \sum_{k=1}^{t} E_k^T P_k(x) E_k,$$

where $P_k(x)$ is an SOS matrix of dimension $|\mathcal{C}_k| \times |\mathcal{C}_k|$.

Proof: apply the **Agler's theorem** to the sparse block matrix Q.

$$P(x) = (I_r \otimes v_d(x))^T \, Q \, (I_r \otimes v_d(x)) = (I_r \otimes v_d(x))^T \left(\sum_{k=1}^{t} E_{\widetilde{\mathcal{C}}_k}^T Q_k E_{\widetilde{\mathcal{C}}_k} \right) (I_r \otimes v_d(x))$$

$$= \sum_{k=1}^{t} \left[(I_r \otimes v_d(x))^T \, E_{\widetilde{\mathcal{C}}_k}^T Q_k E_{\widetilde{\mathcal{C}}_k} \, (I_r \otimes v_d(x)) \right] = \sum_{k=1}^{t} E_{\mathcal{C}_k}^T P_k(x) E_{\mathcal{C}_k},$$

LP/SOCP/SDP

We have the following inclusion relationship

$$DSOS^r_{n,2d}(\mathcal{E},0) \subseteq SDSOS^r_{n,2d}(\mathcal{E},0) \subseteq SSOS^r_{n,2d}(\mathcal{E},0) \subseteq SOS^r_{n,2d}(\mathcal{E},0) \subseteq \mathcal{P}^r_{n,2d}(\mathcal{E},0)$$

Key idea: if a matrix Q is (scaled) diagonally dominant, then it is still (scaled) diagonally dominant when replacing any off-diagonal elements with zeros.

- A brief summary (scalability):

$$\begin{aligned}
\mathcal{P}^r_{n,2d}(\mathcal{E},0) &\longrightarrow \text{NP-hard} \\
DSOS^r_{n,2d}(\mathcal{E},0) &\longrightarrow \text{LP (PSD cones: } 1\times 1) \\
SDSOS^r_{n,2d}(\mathcal{E},0) &\longrightarrow \text{SOCP (PSD cones: } 2\times 2) \\
SSOS^r_{n,2d}(\mathcal{E},0) &\longrightarrow \textbf{SDP with smaller PSD cones of } k\times k \\
SOS^r_{n,2d}(\mathcal{E},0) &\longrightarrow \text{SDP with a PSD cone of } N\times N
\end{aligned}$$

Solution quality: $\mathcal{P}_{\text{dsos}}, \mathcal{P}_{\text{sdsos}}$ and $\mathcal{P}_{\text{ssos}}$ are a sequence of inner approximations with increasing accuracy to the SOS problem $\mathcal{P}_{\text{sos}}$, meaning that

$$f^*_{\text{dsos}} \geq f^*_{\text{sdsos}} \geq f^*_{\text{ssos}} \geq f^*_{\text{sos}},$$

- Similar results can be shown for scalar sparse SOS optimization, which rely on the notion of *correlative sparsity pattern* (Waki *et al.*, 2006).

Implementations and numerical comparison

Packages

- SOS optimization: SOSTOOLS, YALMIP
- DSOS/SDSOS optimization: SPOTLESS
- Chordal decomposition: YALMIP (we adapted the option of correlative sparsity technique)
- SDP solver: Mosek

Numerical examples and applications

- Polynomial optimization problems
- Copositive optimization
- Control application: finding Lyapunov functions

Example 1: Polynomial optimization problems

Eigenvalue bounds on matrix polynomials

$$\begin{array}{rl} \min\limits_{\gamma} & \gamma \\ \text{subject to} & P(x) + \gamma I \succeq 0, \end{array}$$

where $n = 2, 2d = 2$, the polynomial is randomly generated. $P(x)$ has an arrow pattern.

Table: CPU time (in seconds) required by Mosek

Dimension r	10	20	30	40	50	60	70	80
SOS	0.30	1.33	6.64	27.3	108.1	308.7	541.3	1 018.6
SSOS	0.34	0.34	0.35	0.35	0.33	0.32	0.32	0.33
SDSOS	0.47	0.63	1.09	1.29	2.67	3.70	4.40	6.02
DSOS	**	**	**	**	**	**	**	**

**: The program is infeasible.

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Eigenvalue bounds on matrix polynomials

$$\begin{array}{rl} \min_{\gamma} & \gamma \\ \text{subject to} & P(x) + \gamma I \succeq 0, \end{array}$$

where $n = 2, 2d = 2$, the polynomial is randomly generated. $P(x)$ has an arrow pattern.

Table: Optimal value γ

Dimension r	10	20	30	40	50	60	70	80
SOS	1.447	4.813	5.917	4.154	21.61	10.09	7.364	10.19
SSOS	1.454	4.878	5.917	4.498	21.64	12.71	7.558	11.39
SDSOS	40.1	279.3	1 254.4	145.5	762.8	1 521.1	1 217.3	598.0
DSOS	**	**	**	**	**	**	**	**

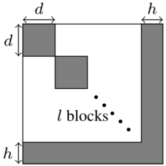
**: The program is infeasible.

Example 2: Copositive optimization

Consider the following copositive program

$$\begin{aligned} \min_{\gamma} \quad & \gamma \\ \text{subject to} \quad & Q + \gamma I \in \mathcal{C}^n, \end{aligned}$$

where Q is a random symmetric matrix with a block-arrow sparsity pattern.



Numerical results

In the simulation, the block size is $d = 3$; arrow head is $h = 2$; we vary the number of blocks l

Table: CPU time (in seconds) required by Mosek

l	2	4	6	8	10
SOS	0.45	7.34	248.9	*	*
SSOS	0.39	0.41	0.38	0.49	0.40
SDSOS	0.54	1.22	4.99	11.07	32.18
DSOS	0.59	0.76	2.19	5.72	17.11

*: Out of memory.

Example 2: Copositive optimization

Consider the following copositive program

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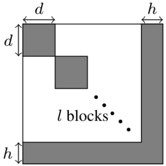


Numerical results

In the simulation, the block size is $d = 3$; arrow head is $h = 2$; we vary the number of blocks l

Table: Optimal value γ

l	2	4	6	8	10
SOS	1.137	4.197	2.836	*	*
SSOS	1.137	4.197	2.836	4.043	4.718
SDSOS	1.184	4.500	3.282	4.562	5.146
DSOS	2.551	7.775	6.452	12.057	15.203

*: Out of memory.

Example 3: Finding Lyapunov functions

Control application: finding Lyapunov functions

- Consider a dynamical system with a banded pattern

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), & g_1(x) &= \gamma - x_1^2 \geq 0 \\ \dot{x}_2 &= f_2(x_1, x_2, x_3), & g_2(x) &= \gamma - x_2^2 \geq 0 \\ &\vdots \\ \dot{x}_n &= f_n(x_{n-1}, x_n), & g_2(x) &= \gamma - x_n^2 \geq 0 \end{aligned}$$

- Generate locally stable systems of degree three;
- Consider a polynomial Lyapunov function of degree two with a banded pattern

$$V(x) = V_1(x_1, x_2) + V_2(x_1, x_2, x_3) + \ldots + V_n(x_{n-1}, x_n)$$

- Then, we consider the following SOS program

$$\begin{aligned} \text{Find} \quad & V(x), r_i(x) \\ \text{subject to} \quad & V(x) - \epsilon(x^T x) \text{ is SOS} \\ & -\langle \nabla V(x), f(x) \rangle - \sum_{i=1}^{n} r_i(x) g_i(x) \text{ is SOS} \\ & r_i(x) \text{ is SOS}, i = 1, \ldots, n. \end{aligned}$$

Example 3: Finding Lyapunov functions

Control application: finding Lyapunov functions

Table: CPU time (in seconds) required by Mosek

n	10	15	20	30	40	50
SOS	1.29	18.44	247.84	*	*	*
SSOS	0.55	0.68	0.71	0.83	1.04	1.17
SDSOS	0.71	1.76	4.47	32.21	85.99	257.20
DSOS	0.70	1.42	3.58	35.12	73.64	324.32

*: Out of memory.

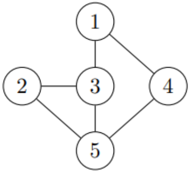
Part III - Beyond chordal decomposition

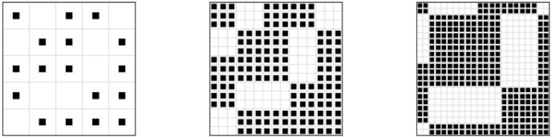
Extension 1: Block chordal decomposition

Classical chordal decomposition:

$$\underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}$$

where $*$ denotes a **real scalar number**.
Question: Does the decomposition still hold if $*$ denotes a block of arbitrary size?





(a) (b) (c)

Figure: Sparse block matrices: (a) $\alpha_1 = \{1, 1, 1, 1, 1\}$, (b) $\alpha_2 = \{3, 3, 3, 3, 3\}$, (c) $\alpha_3 = \{2, 8, 4, 6, 2\}$

Theorem: Both the decomposition and completion results hold for **sparse block matrices** with a chordal pattern!

Extension 1: Block chordal decomposition

Theorem: Both the decomposition and completion results hold for **sparse block matrices** with a chordal pattern!

$$\underbrace{\left[\begin{array}{c|c|c} * & * & 0 \\ \hline * & * & * \\ \hline 0 & * & * \end{array}\right]}_{\succeq 0} = \underbrace{\left[\begin{array}{c|c|c} * & * & 0 \\ \hline * & * & 0 \\ \hline 0 & 0 & 0 \end{array}\right]}_{\succeq 0} + \underbrace{\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & * & * \\ \hline 0 & * & * \end{array}\right]}_{\succeq 0}.$$

$$\underbrace{\left[\begin{array}{cc|c|cc} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline * & * & * & * & * \\ \hline 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array}\right]}_{\succeq 0} = \underbrace{\left[\begin{array}{cc|c|cc} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ \hline * & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\succeq 0} + \underbrace{\left[\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & * \\ \hline 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{array}\right]}_{\succeq 0},$$

$$\underbrace{\left[\begin{array}{cc|cc|cc} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ \hline * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right]}_{\succeq 0} = \underbrace{\left[\begin{array}{cc|cc|cc} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ \hline * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\succeq 0} + \underbrace{\left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ \hline 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right]}_{\succeq 0}.$$

Extension 2: PSD polynomial matrices

Question: Decomposition of PSD polynomial matrices?

$$P(x) = \begin{bmatrix} p_{11}(x) & p_{12}(x) & \\ p_{21}(x) & p_{22}(x) & p_{23}(x) \\ & p_{32}(x) & p_{33}(x) \end{bmatrix} \succeq 0, \forall x \in \mathbb{R}^n.$$

Restriction to SOS matrix: $P(x) \in SSOS^r_{n,2d}(\mathcal{E}, 0) \Leftrightarrow P(x) = \sum_{k=1}^{t} E_k^T P_k(x) E_k$, where $P_k(x)$ is an SOS matrix.

Generalization: The same chordal decomposition result holds if $P_k(x)$ is allowed to have rational function entries.

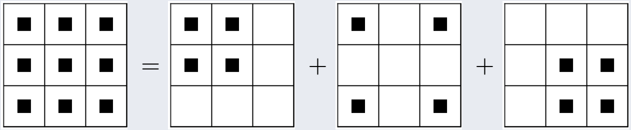
$$\underbrace{\begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0} = \underbrace{\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\succeq 0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}}_{\succeq 0}.$$

where $*$ denotes a rational function (i.e., a ratio of two polynomials).

Extension 3: Block factor-width two matrices

Question: How to deal with dense PSD matrices with no chordal sparsity?
Answer: One useful solution is to use an inner approximation of the PSD cone, *e.g.*, DD and SDD matrices.

(Block) factor-width two matrices via a block partition



- A new hierarchy of inner approximations of the PSD cone by varying the block partition.

  Given three partitions $\gamma = \{k_1, \ldots, k_p\}, \beta = \{l_1, \ldots, l_p\}$ and $\alpha = \{n_1, n_2\}$, where $\sum_{i=1}^{p} k_i = \sum_{i=1}^{q} l_i = n_1 + n_2 = n$ and $\alpha \sqsupseteq \beta \sqsupseteq \gamma$, we have the following inclusion:

$$SDD = \mathcal{FW}^n_{\mathbf{1},2} \subseteq \mathcal{FW}^n_{\gamma,2} \subseteq \mathcal{FW}^n_{\beta,2} \subseteq \mathcal{FW}^n_{\alpha,2} = \mathbb{S}^n_+$$

Extension 3: Block factor-width two matrices

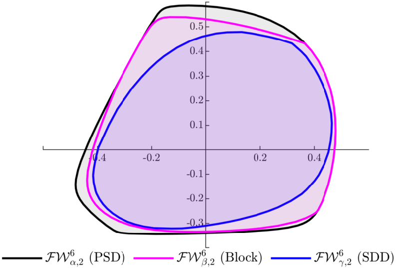


Figure: Boundary of x and y for which the 6×6 symmetric matrix $I_6 + xA + yB$ belongs to $\mathcal{FW}^6_{\alpha,2}, \mathcal{FW}^6_{\beta,2}$, and $\mathcal{FW}^6_{\gamma,2}$, where $\alpha = \{4, 2\}$, $\beta = \{2, 2, 2\}$, $\gamma = \{1, 1, 1, 1, 1, 1\}$.

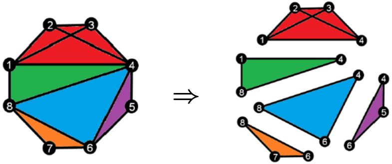
A new method to balance a trade-off between the **computation scalability** and **solution quality**!

- **More examples**: Y. Zheng, A. Sootla, and A. Papachristodoulou. "Block factor-width-two matrices and their applications to semidefinite and sum-of-squares optimization." arXiv:1909.11076 (2019).

Conclusion

Take-home message

- **Message 1: Chordal decomposition:** leading to sparse PSD cone decompositions



- **Message 2: Sparse SDPs can be solved 'fast'**

$$\begin{aligned} \min_{x, x_k} \quad & \langle c, x \rangle \\ \text{s.t.} \quad & Ax = b, \\ & \boxed{x_k = H_k x}, \quad k = 1, \ldots, p, \\ & x_k \in \mathcal{S}_k, \quad k = 1, \ldots, p, \end{aligned}$$

$$P(x) \in SSOS^r_{n,2d}(\mathcal{E}, 0)$$

$$\Longleftrightarrow P(x) = \sum_{k=1}^{t} E_k^T P_k(x) E_k,$$

CDCS: an open-source first-order conic solver;

Download from `https://github.com/OxfordControl/CDCS`

- **Message 3: Sparse SOS optimization can be solved 'fast':** Bridging the gap between DSOS/SDSOS optimization and SOS optimization.

Thank you for your attention!

Q & A

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