Scalable Learning, Optimization, and Control for Autonomous Systems

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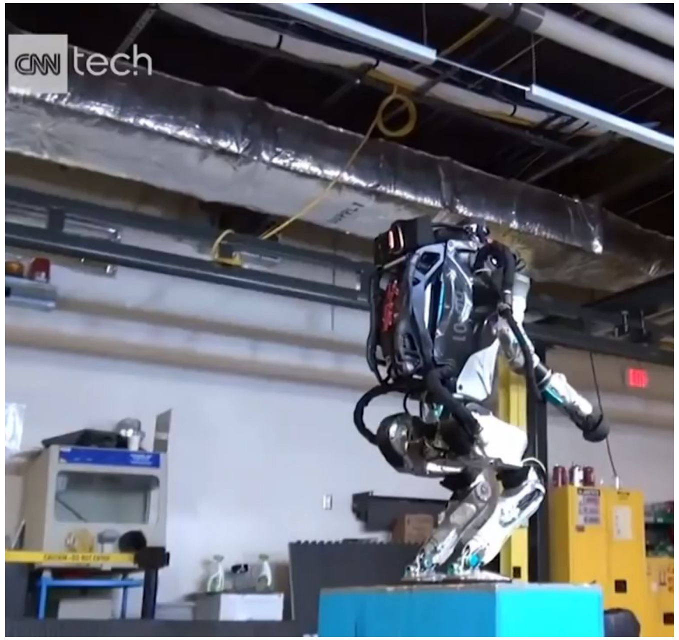
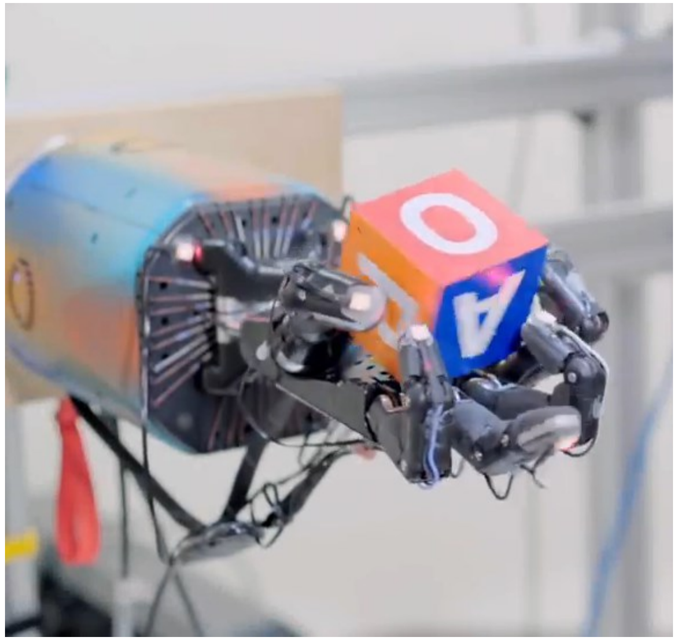
UC San Diego
JACOBS SCHOOL OF ENGINEERING
Electrical and Computer Engineering

Scalable Optimization and Control (SOC) Lab
https://zhengy09.github.io/soclab.html

Successful stories



Boston Dynamics



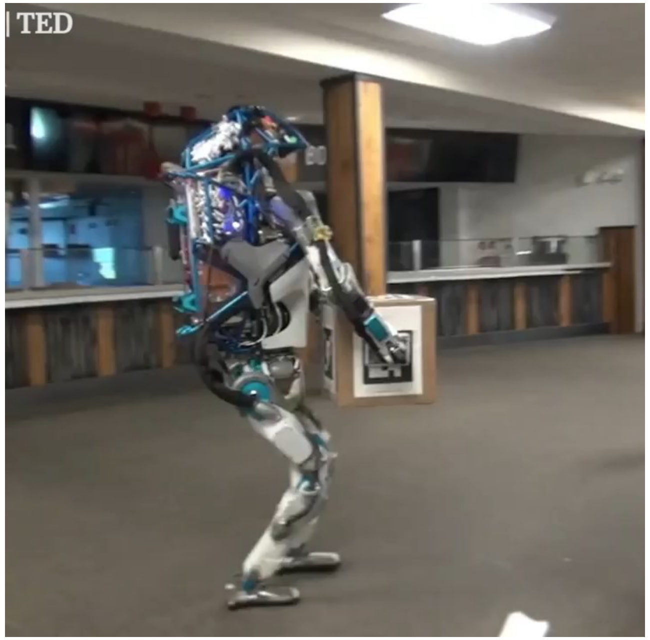
Open AI

Waymo

Learning, Optimization, and Control are fundamental building blocks

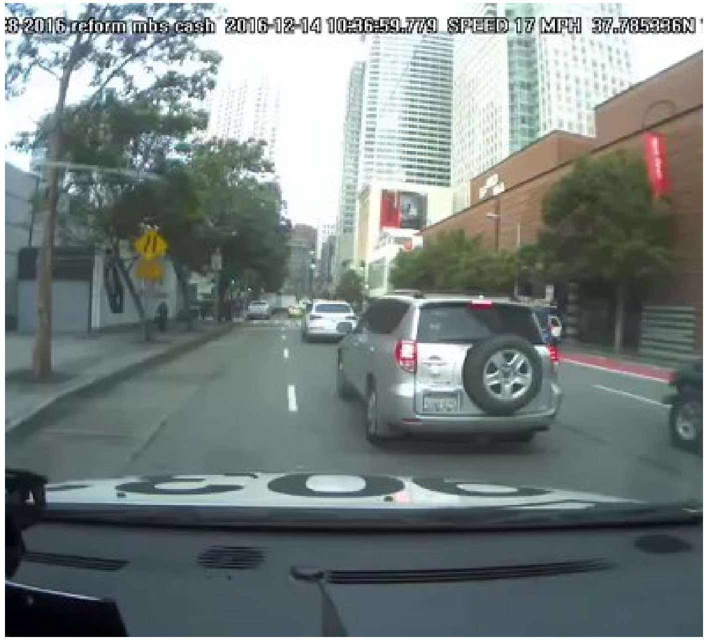
Video from Nikolai Matni's talk

Lots of failure stories



Boston Dynamics

DeepMind



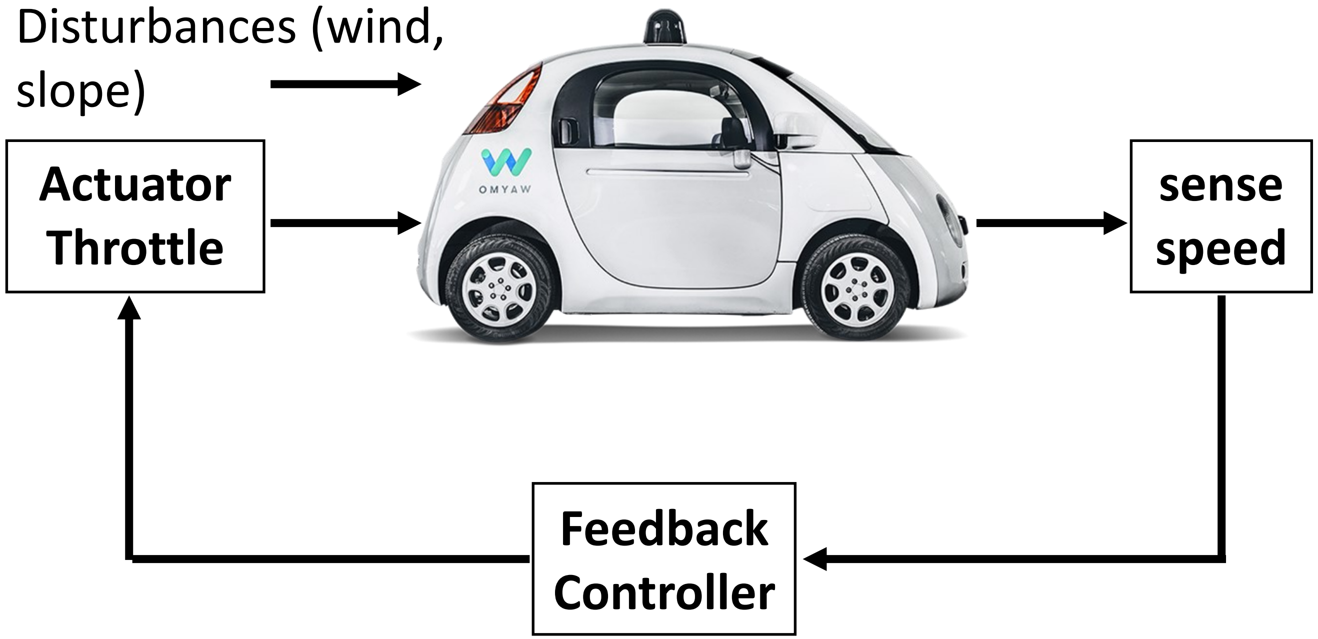
Uber running a red

Formal guarantees of robustness, safety, and efficiency are very challenging

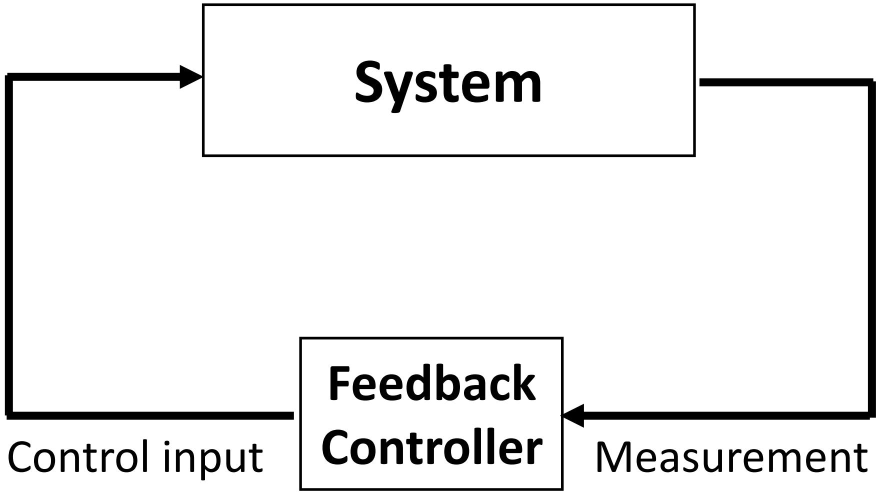
Video from Nikolai Matni’s talk

Automatic control example

Highway cruise control



Feedback Paradigm



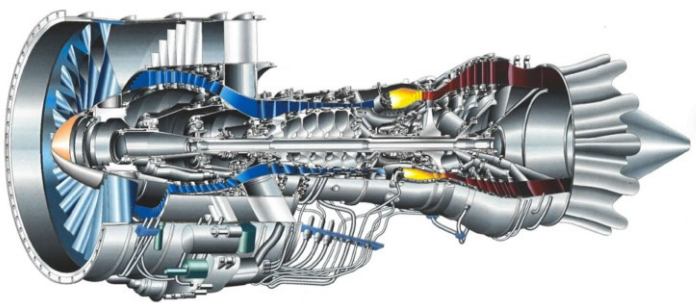
Control theory: the principled use of feedback loops and algorithms to drive a system to its desired goal

- ❑ “Simple” centralized linear control systems are well understood.
- ❑ “Complexity” can enter in different ways . . .

Complex autonomous systems

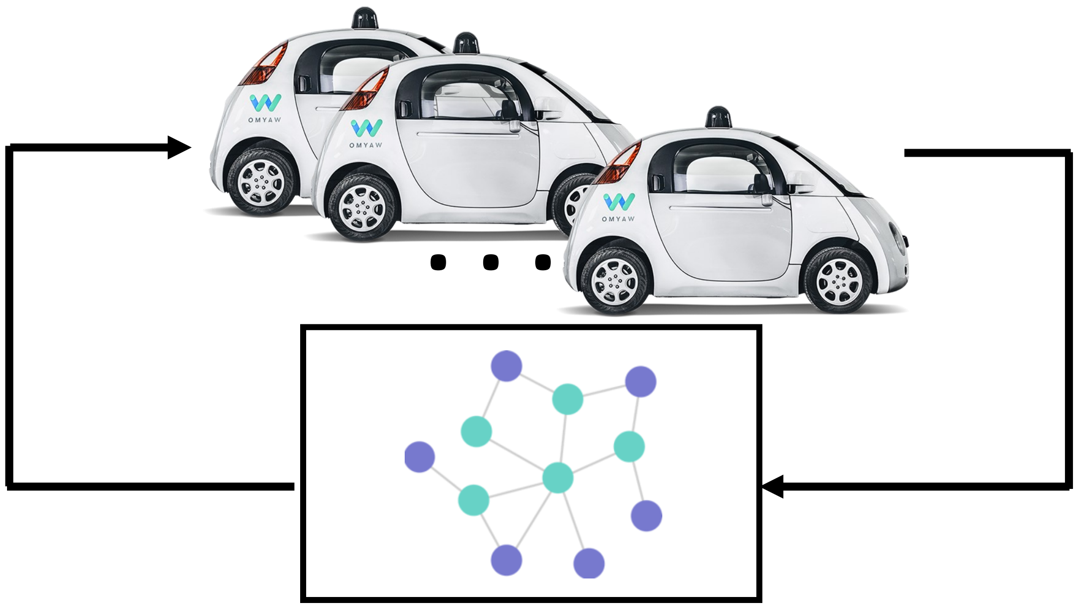
❑ Complex nonlinear dynamics

- Aircraft, jet engine, robotics



❑ Complex distributed systems

- Multiple subsystems & local commutation

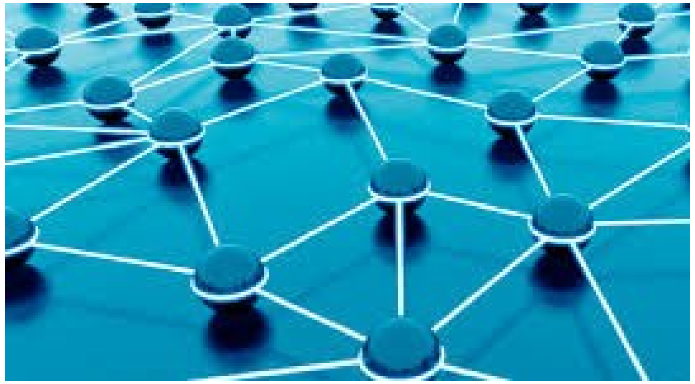


Distributed controller

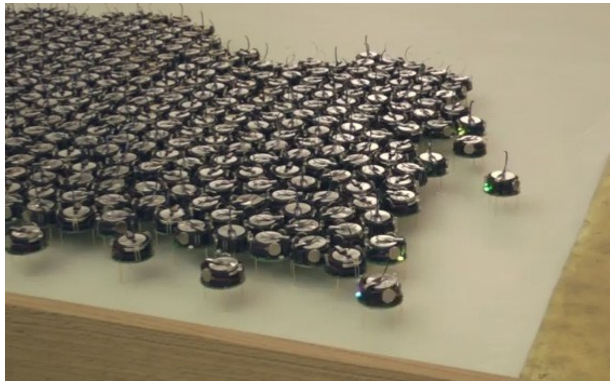
Source: https://solidmechanicsproblems.wordpress.com/; https://www.bostondynamics.com/

Examples of large-scale autonomous systems

Drone formations

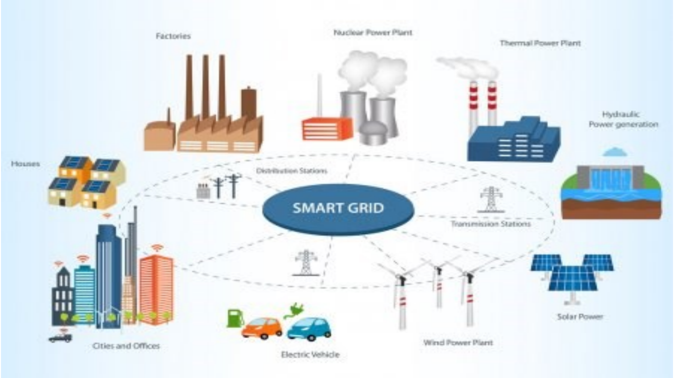


Sensor networks



Robotic networks

Transportation network



Smart grid

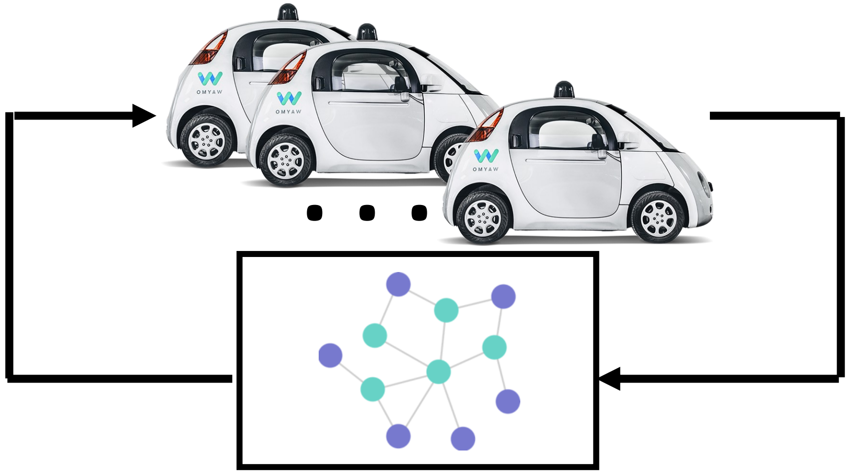
Self-organization

Distributed control laws → ***Desired collective behavior***

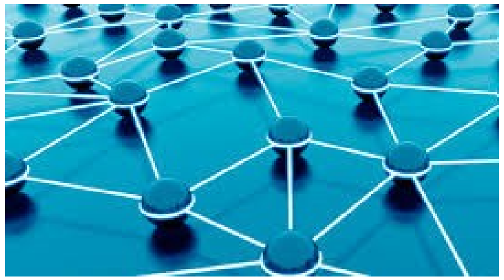
Image Credit: Getty Images; Google Images; www.pinterest.com

Challenges and Overview of the SOC lab

- ❑ **Model uncertainty** ➡ Learning-based & Robust control
  - Model might be unknown for practical systems;
  - Model might be uncertain; **Learning-based** solutions
- ❑ **Information constraints** ➡ Distributed control
  - Large numbers of components;
  - Subsystems or components may have dynamic coupling;
  - Only **local information available** for control decision;
- ❑ **High dimensional problems** ➡ Scalable Optimization
  - A very large number of states and control variables;
  - Require to solve **large-scale optimization** efficiently;
- ❑ **Real world applications** ➡ Mixed traffic control



Distributed controller

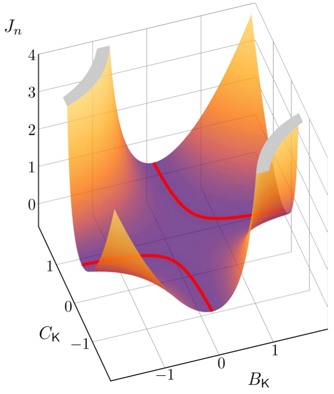


Scalable Optimization & Control (SOC) Lab

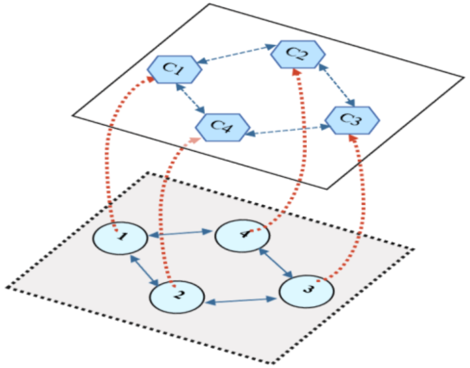
Scalable Optimization and Control (SOC) lab

SOC Lab

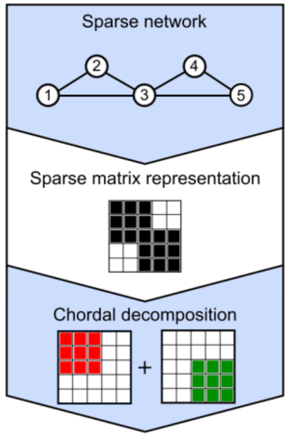
Data-driven and learning-based control



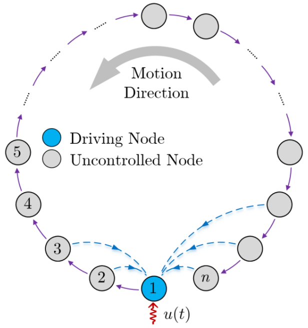
Scalable distributed control



Sparse conic optimization



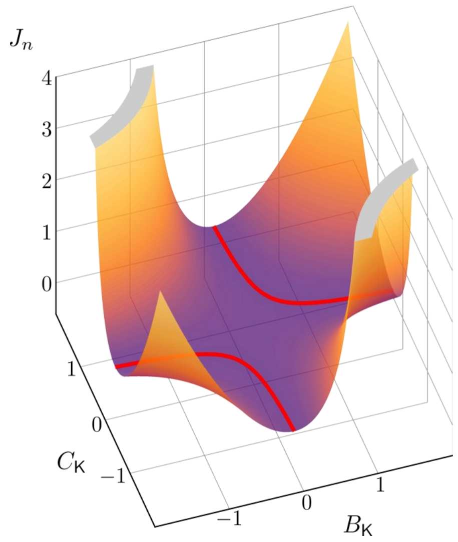
Connected and autonomous vehicles (CAVs)



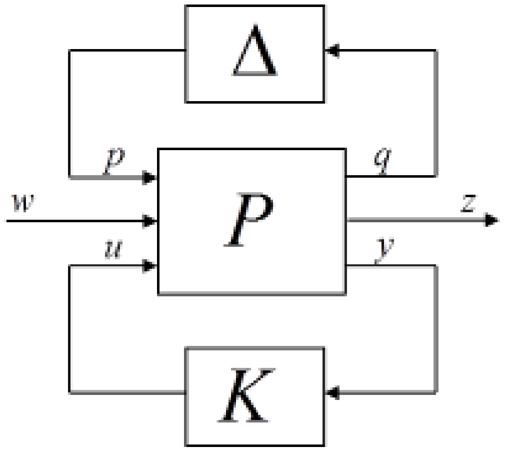
Check out our webpage: https://zhengy09.github.io/soclab.html

Today's talk

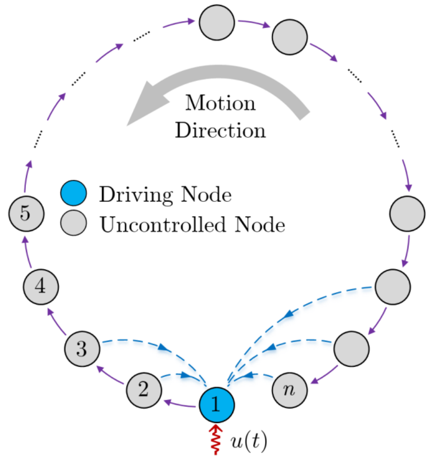
Part 1: Model free LQG control



Part 2: Model-based LQG control



Part 3: Data-driven MPC in mixed traffic



Analysis of the Optimization Landscape of Linear Quadratic Gaussian (LQG) Control

Yujie Tang
Harvard University

Na Li
Harvard University

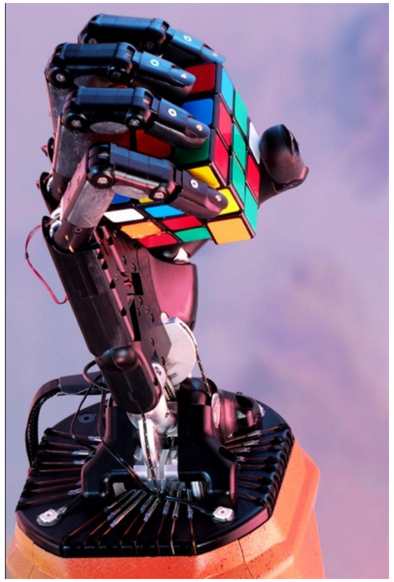
Zheng, Y., Tang, Y., & Li, N. (2021). Analysis of the optimization landscape of linear quadratic gaussian (LQG) control. arXiv preprint arXiv:2102.04393.

Motivation

❑ Model-free methods and data-driven control

- Use direct policy updates
- Become very popular in both academia and practice, from game playing, robotics, and drones, etc.

DeepMind



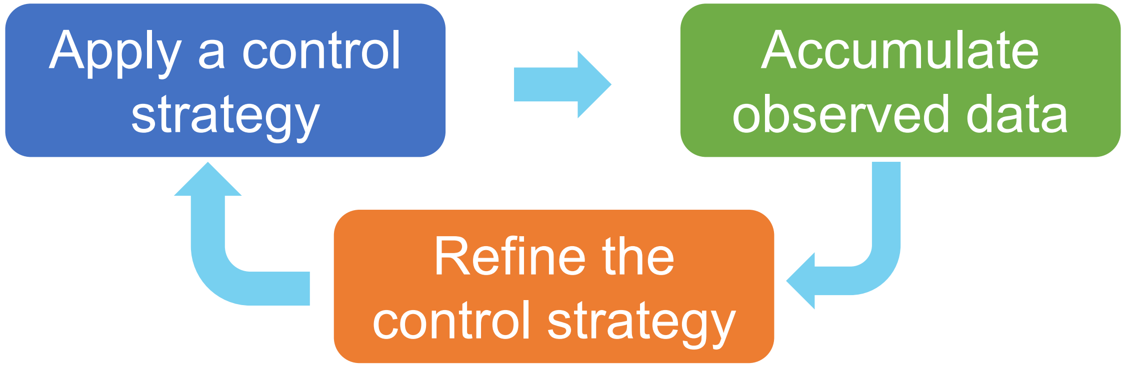
OpenAI

Applications

Duan et al. 2016; Silver et al., 2017; Dean et al., 2019; Tu and Recht, 2019;
Mania et al., 2019; Fazel et al., 2018; Recht, 2019;

Motivation

❑ Model-free methods and data-driven control



Opportunities

- Directly search over a given policy class
- Directly optimize performance on the true system, bypassing the model estimation (not on an approximated model)

Challenges

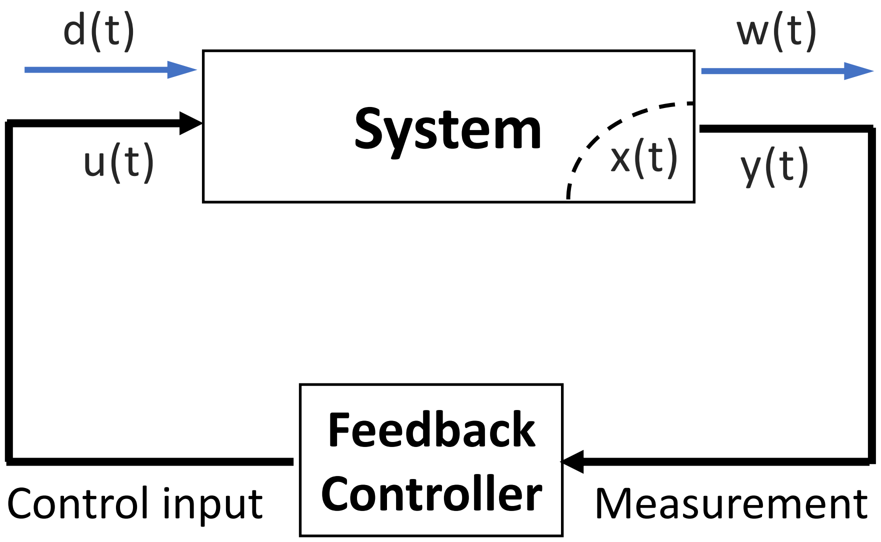
- Lack of non-asymptotic performance guarantees
  - ➢ Sample complexity
  - ➢ Suboptimality
  - ➢ Convergence, etc.

❖ Highly nontrivial even for **linear dynamical systems**

Today's talk

❑ Optimal Control

Feedback Paradigm



Linear Quadratic Optimal control

$$\min_{u_1,u_2,\ldots,} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^{\mathsf{T}} Q x_t + u_t^{\mathsf{T}} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = Ax_t + Bu_t + w_t$$

$$y_t = Cx_t + v_t$$

- Many practical applications
- **Linear Quadratic Regulator (LQR)** when the state x_t is directly observed
- **Linear Quadratic Gaussian (LQG) control** when only partial output y_t is observed
- Extensive classical results (Dynamic programming, Separation principle, Riccati equations, etc.)

Major challenge: how to perform optimal control when the system is unknown?

Model-free: Direct policy iteration

❑ Controller parameterization

- Give a parameterization of control policies; say **neural networks?** ✖
- Control theory already tells us many structural properties
- **Linear feedback is sufficient for LQR** $u_t = Kx_t$

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_t^{\mathsf{T}}Qx_t + u_t^{\mathsf{T}}Ru_t\right)\right] := J(K)$$

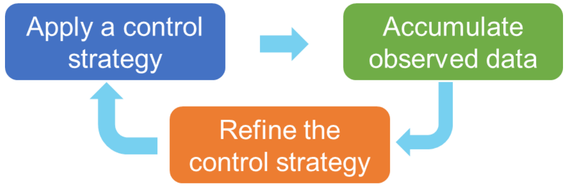
- Set of stabilizing controllers $K \in \mathcal{K}$

- A fast-growing list of references

LQR as an Optimization problem

$$\min_{K} \quad J(K)$$
$$\text{s.t.} \quad K \in \mathcal{K}$$

Direct policy iteration



$$K_{i+1} = K_i - \alpha_i \nabla J(K_i)$$

✓ Good Landscape properties (Fazel et al., 2018)
- Connected feasible region
- Unique stationary point
- Gradient dominance

✓ Fast global convergence (exponential)

➢ Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others

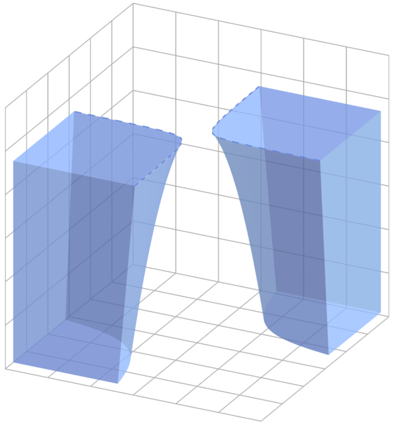
Challenges for partially observed LQG

❑ Results on model-free LQG control are much fewer

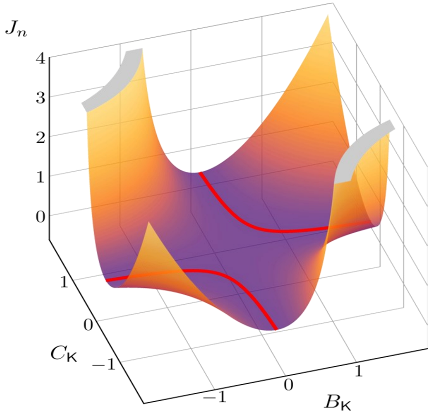
- LQG is more complicated than LQR
- Requires dynamical controllers
- Its landscape properties are much richer and more complicated than LQR

Our focus: Landscape Analysis of LQG

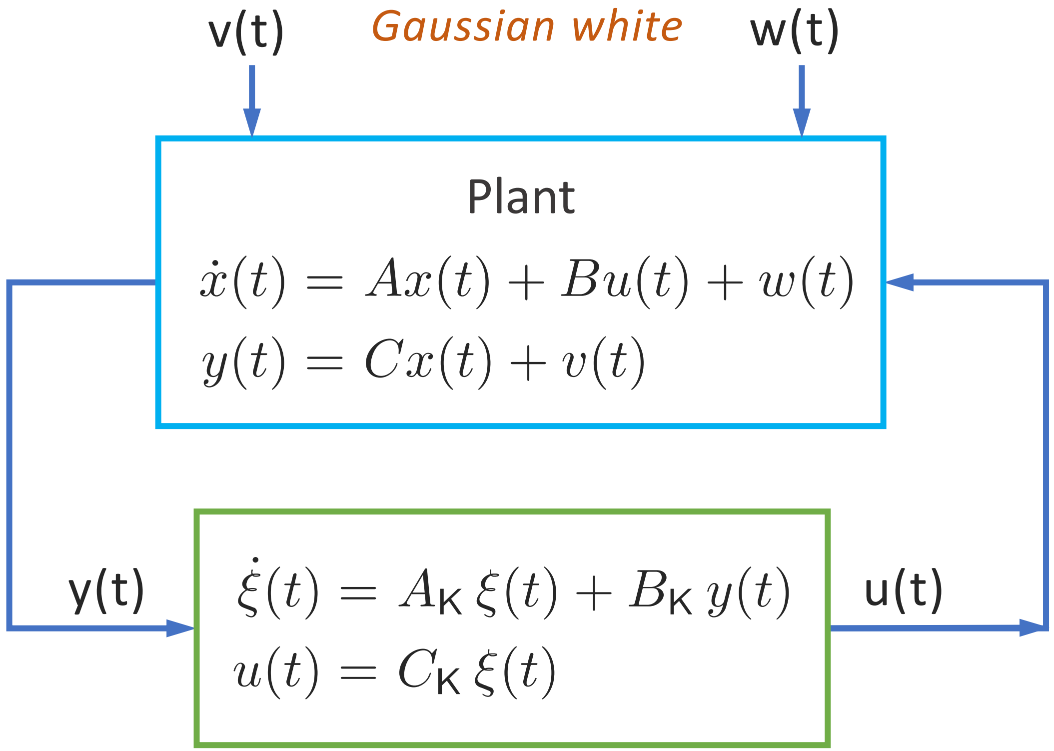
- **Question 1: Properties of the domain** (set of stabilizing controllers)
  - convexity, connectivity, open/closed?



- **Question 2: Properties of the accumulated cost**
  - convexity, differentiability, coercivity?
  - set of stationary points/local minima/global minima?



LQG Problem Setup



dynamical controller

$\mathsf{K} = (A_\mathsf{K}, B_\mathsf{K}, C_\mathsf{K})$

Standard Assumption	(A, B), $(A, W^{1/2})$	Controllable
	(C, A), $(Q^{1/2}, A)$	Observable

Objective: The LQG cost

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T (x^\top Q x + u^\top R u)\, dt$$

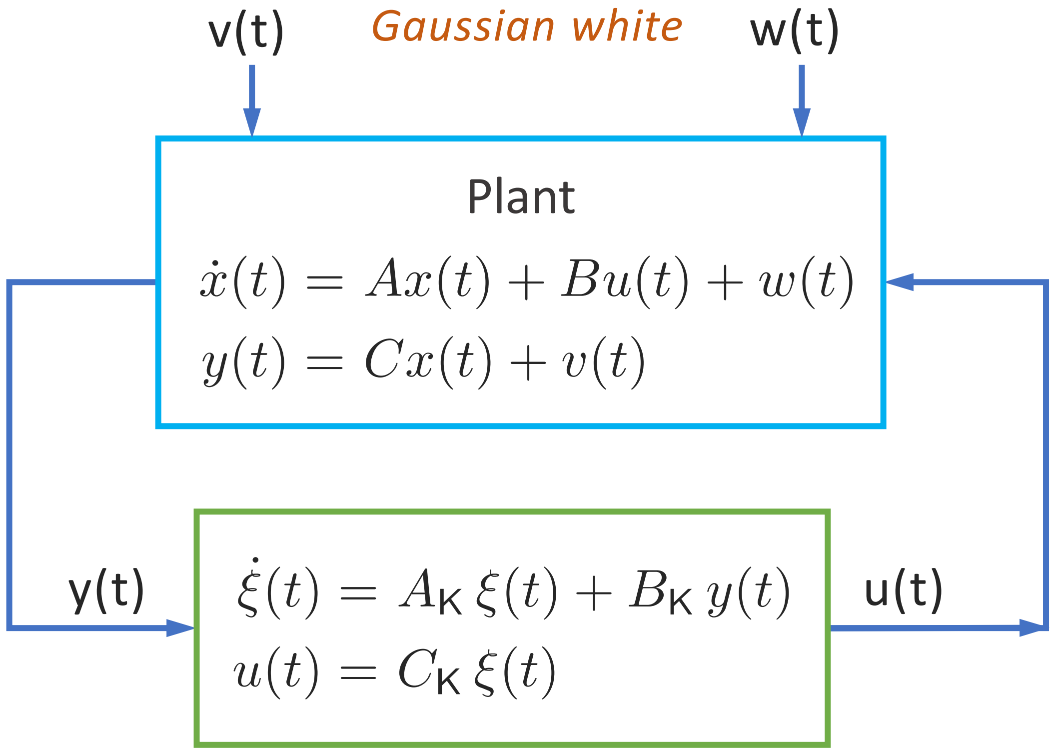
- $\xi(t)$ internal state of the controller
- $\dim \xi(t)$ order of the controller
- $\dim \xi(t) = \dim x(t)$ full-order
- $\dim \xi(t) < \dim x(t)$ reduced-order

Minimal controller

The input-output behavior cannot be replicated by a lower order controller.

* $(A_\mathsf{K}, B_\mathsf{K}, C_\mathsf{K})$ controllable and observable

Separation principle



dynamical controller

$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$

Explicit dependence on the dynamics

Objective: The LQG cost

$$\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T (x^\top Q x + u^\top R u)\, dt$$

Solution: Kalman filter for state estimation
+ LQR based on the estimated state

$$\dot{\xi} = (A - BK)\xi + L(y - C\xi),$$
$$u = -K\xi.$$

Two Riccati equations

- **Kalman gain** $L = PC^{\mathsf{T}}V^{-1}$

$$AP + PA^{\mathsf{T}} - PC^{\mathsf{T}}V^{-1}CP + W = 0,$$

- **Feedback gain** $K = R^{-1}B^{\mathsf{T}}S$

$$A^{\mathsf{T}}S + SA - SBR^{-1}B^{\mathsf{T}}S + Q = 0$$

Model-free Optimization formulation

Closed-loop dynamics

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & B_{\mathsf{K}} \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C_{\mathsf{K}} \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}$$

Feasible region of the controller parameters

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} \;\middle|\; \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \text{ is full-order}, \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \text{ is Hurwitz stable} \right\}$$

Cost function $\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T (x^\top Q x + u^\top R u)\, dt$

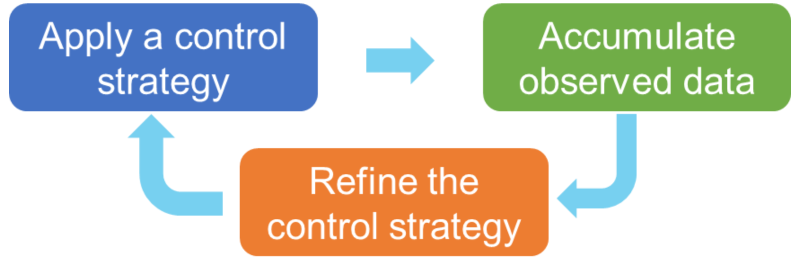
$$J(\mathsf{K}) = \operatorname{tr}\left(\begin{bmatrix} Q & 0 \\ 0 & C_{\mathsf{K}}^{\top} R C_{\mathsf{K}} \end{bmatrix} X_{\mathsf{K}}\right) = \operatorname{tr}\left(\begin{bmatrix} W & 0 \\ 0 & B_{\mathsf{K}} V B_{\mathsf{K}}^{\top} \end{bmatrix} Y_{\mathsf{K}}\right)$$

$X_{\mathsf{K}}, Y_{\mathsf{K}}$ Solution to Lyapunov equations

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$
$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

Direct policy iteration $\mathsf{K}_{i+1} = \mathsf{K}_i - \alpha_i \nabla J(\mathsf{K}_i)$

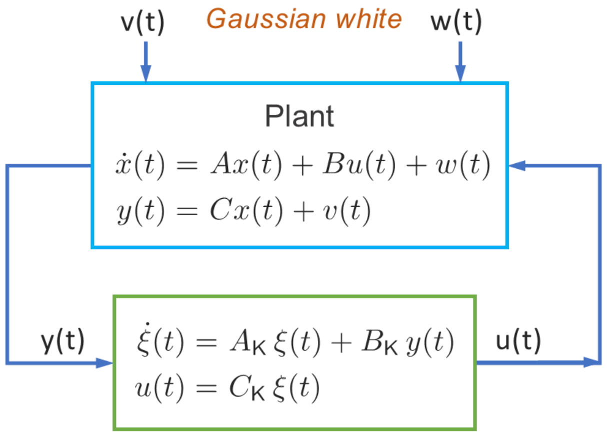


- ✓ **Does it converge at all?**
- ✓ **Converge to which point?**
- ✓ **Convergence speed?**

Landscape Analysis

Hyland, David, and Dennis Bernstein. "The optimal projection equations for fixed-order dynamic compensation." *IEEE Transactions on Automatic Control* 29.11 (1984): 1034-1037.

Model-free Optimization formulation



dynamical controller
$\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}})$

Landscape Analysis

LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$
$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

- **Q1: Connectivity of the feasible region** $\mathcal{C}_{\text{full}}$
  - Is it connected?
  - If not, how many connected components can it have?
- **Q2: Structure of stationary points of** $J(\mathsf{K})$
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?

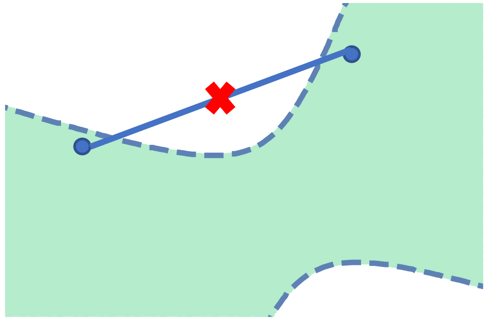
Connectivity of the feasible region

❑ Simple observation: non-convex and unbounded

Lemma 1: the set $\mathcal{C}_{\text{full}}$ is non-empty, unbounded, and can be non-convex.

Example

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$



$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \middle| \begin{bmatrix} 1 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \text{ is stable} \right\}.$$

$$\mathsf{K}^{(1)} = \begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}, \qquad \mathsf{K}^{(2)} = \begin{bmatrix} 0 & -2 \\ 2 & -2 \end{bmatrix}$$

Stabilize the plant, and thus belong to $\mathcal{C}_{\text{full}}$

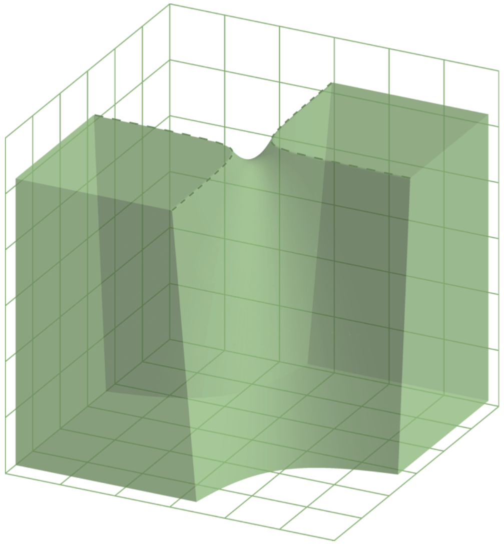
$$\hat{\mathsf{K}} = \frac{1}{2}\left(\mathsf{K}^{(1)} + \mathsf{K}^{(2)}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

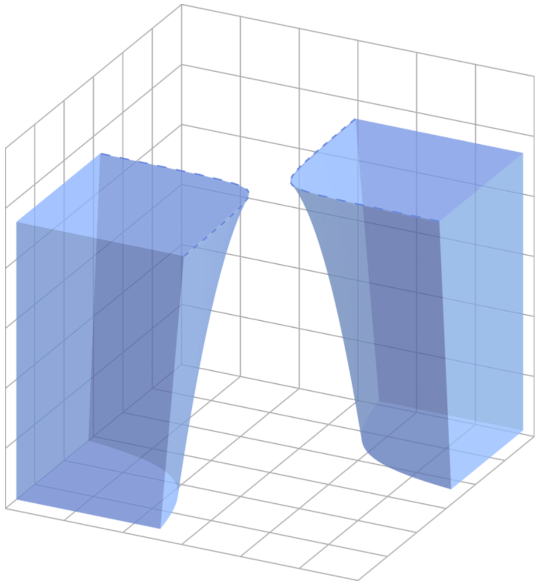
Fails to stabilize the plant, and thus outside $\mathcal{C}_{\text{full}}$

Connectivity of the feasible region

❑ Main Result 1: dis-connectivity

Theorem 1: The set $\mathcal{C}_{\text{full}}$ can be disconnected but has at most 2 connected components.



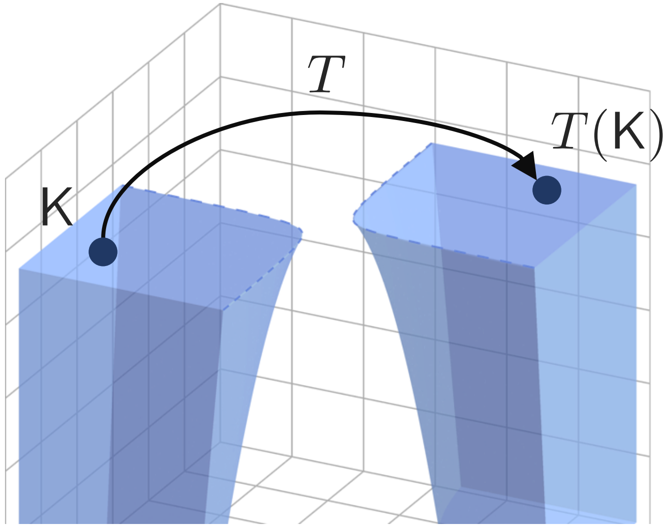


- ✓ Different from the connectivity of static stabilizing state-feedback controllers, which is always connected!
- ✓ **Is this a negative result for gradient-based algorithms? →** No

Connectivity of the feasible region

❑ Main Result 2: dis-connectivity

Theorem 2: If $\mathcal{C}_{\text{full}}$ has 2 connected components, then there is a smooth bijection T between the 2 connected components that has the same cost function value.



$$J(\mathsf{K}) = J(T(\mathsf{K}))$$

✓ In fact, the bijection T is defined by a similarity transformation (change of controller state coordinate)

$$\mathscr{T}_T(\mathsf{K}) := \begin{bmatrix} D_{\mathsf{K}} & C_{\mathsf{K}}T^{-1} \\ TB_{\mathsf{K}} & TA_{\mathsf{K}}T^{-1} \end{bmatrix}.$$

Positive news: For gradient-based local search methods, it makes no difference to search over either connected component.

Connectivity of the feasible region

❑ Main Result 3: conditions for connectivity

Theorem 3: 1) $\mathcal{C}_{\text{full}}$ is connected if there exists a reduced-order stabilizing controller.

2) The sufficient condition above becomes necessary if the plant is single-input or single-output.

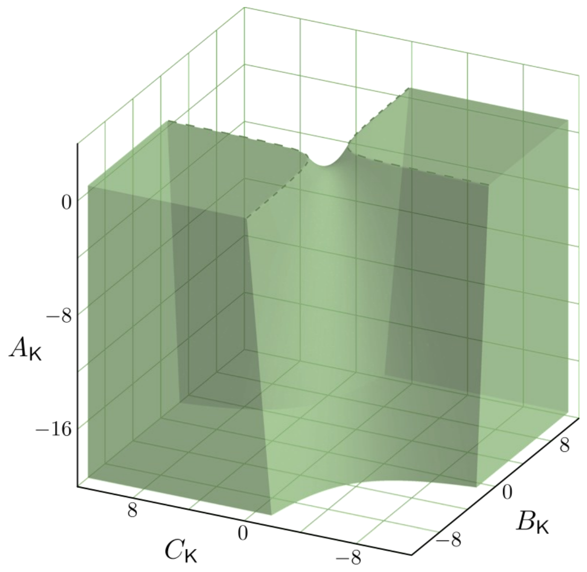
Corollary 1: Given any open-loop stable plant, the set of stabilizing controllers $\mathcal{C}_{\text{full}}$ is connected.

Example: Open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

Routh--Hurwitz stability criterion

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \middle| A_{\mathsf{K}} < 1, B_{\mathsf{K}} C_{\mathsf{K}} < -A_{\mathsf{K}} \right\}.$$



Connectivity of the feasible region

❑ Main Result 3: conditions for connectivity

Example: Open-loop unstable system (SISO)

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

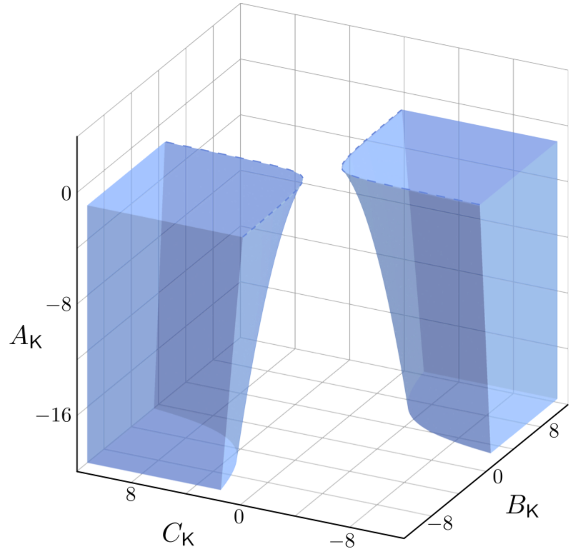
- **Routh--Hurwitz stability criterion**

$$\mathcal{C}_{\text{full}} = \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, \begin{bmatrix} A & BC_{\mathsf{K}} \\ B_{\mathsf{K}}C & A_{\mathsf{K}} \end{bmatrix} \text{ is stable} \right\}$$
$$= \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_{\mathsf{K}} < -1,\ B_{\mathsf{K}}C_{\mathsf{K}} < A_{\mathsf{K}} \right\}.$$

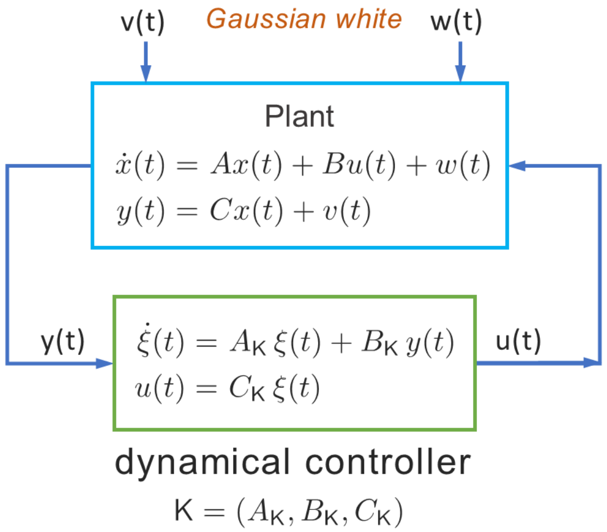
- **Two path-connected components**

$$\mathcal{C}_1^+ := \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_{\mathsf{K}} < -1,\, B_{\mathsf{K}}C_{\mathsf{K}} < A_{\mathsf{K}},\, B_{\mathsf{K}} > 0 \right\},$$
$$\mathcal{C}_1^- := \left\{ \mathsf{K} = \begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, A_{\mathsf{K}} < -1,\, B_{\mathsf{K}}C_{\mathsf{K}} < A_{\mathsf{K}},\, B_{\mathsf{K}} < 0 \right\}.$$

Disconnected feasible region



Model-free Optimization formulation



LQG as an Optimization problem

$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$

$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

Landscape Analysis

- **Q1: Connectivity of the feasible region $\mathcal{C}_{\text{full}}$**
  - Is it connected? **No**
  - How many connected components can it have? **Two**
- **Q2: Structure of stationary points of $J(\mathsf{K})$**
  - Are there spurious (strictly suboptimal, saddle) stationary points?
  - How to check if a stationary point is globally optimal?

Structure of Stationary Points

❑ Simple observations

1) $J(\mathsf{K})$ is a real analytic function over its domain (smooth, infinitely differentiable)
2) $J(\mathsf{K})$ has **non-unique** and **non-isolated** global optima

LQG as an Optimization problem

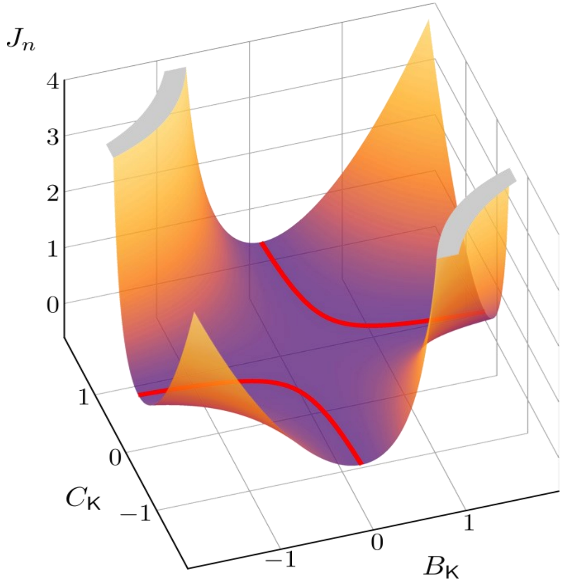
$$\min_{\mathsf{K}} \quad J(\mathsf{K})$$
$$\text{s.t.} \quad \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$$

$$\dot{\xi}(t) = A_{\mathsf{K}}\,\xi(t) + B_{\mathsf{K}}\,y(t)$$
$$u(t) = C_{\mathsf{K}}\,\xi(t)$$

Similarity transformation

$$(A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \mapsto (TA_{\mathsf{K}}T^{-1}, TB_{\mathsf{K}}, C_{\mathsf{K}}T^{-1})$$

- $J(\mathsf{K})$ is invariant under similarity transformations.
- It has many stationary points, unlike the LQR with a unique stationary point



Structure of Stationary Points

❑ Gradient computation

Lemma 1: For every $\mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$, we have

$$\frac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 2\left(Y_{12}^{\mathsf{T}} X_{12} + Y_{22} X_{22}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 2\left(Y_{22} B_{\mathsf{K}} V + Y_{22} X_{12}^{\mathsf{T}} C^{\mathsf{T}} + Y_{12}^{\mathsf{T}} X_{11} C^{\mathsf{T}}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 2\left(R C_{\mathsf{K}} X_{22} + B^{\mathsf{T}} Y_{11} X_{12} + B^{\mathsf{T}} Y_{12} X_{22}\right),$$

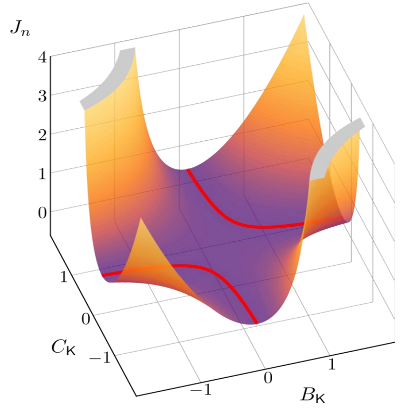
where $X_{\mathsf{K}} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22} \end{bmatrix}, \qquad Y_{\mathsf{K}} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathsf{T}} & Y_{22} \end{bmatrix}$

are the unique positive semidefinite solutions to two Lyapunov equations.

How does the set of Stationary Points look like?

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \frac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 0,\ \frac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 0,\ \frac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 0, \right\}$$

- ❑ Non-unique, non-isolated
- ❑ Local minimum, local maximum, saddle points, or globally minimum?



Structure of Stationary Points

Main Result: existences of strict saddle points

Theorem 4: Consider any open-loop stable plant. The zero controller with any stable A_{K}

$$\mathsf{K} = (A_{\mathsf{K}}, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero.

Example:

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$

$$Q = 1, R = 1, V = 1, W = 1$$

Stationary point: $\mathsf{K}^\star = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \in \mathbb{R}^{2\times 2}, \qquad \text{with } a < 0$

- **Cost function:** $J\left(\begin{bmatrix} 0 & C_{\mathsf{K}} \\ B_{\mathsf{K}} & A_{\mathsf{K}} \end{bmatrix}\right) = \dfrac{A_{\mathsf{K}}^2 - A_{\mathsf{K}}(1 + B_{\mathsf{K}}^2 C_{\mathsf{K}}^2) - B_{\mathsf{K}} C_{\mathsf{K}}(1 - 3B_{\mathsf{K}} C_{\mathsf{K}} + B_{\mathsf{K}}^2 C_{\mathsf{K}}^2)}{2(-1 + A_{\mathsf{K}})(A_{\mathsf{K}} + B_{\mathsf{K}} C_{\mathsf{K}})}.$

- **Hessian:** $$\begin{bmatrix} \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}}^2} & \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}} \partial B_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial A_{\mathsf{K}} \partial C_{\mathsf{K}}} \\ \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}} A_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}}^2} & \frac{\partial J^2(\mathsf{K})}{\partial B_{\mathsf{K}} \partial C_{\mathsf{K}}} \\ \frac{\partial J^2(\mathsf{K})}{\partial C_{\mathsf{K}} A_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial C_{\mathsf{K}} B_{\mathsf{K}}} & \frac{\partial J^2(\mathsf{K})}{\partial \partial C_{\mathsf{K}}^2} \end{bmatrix}\Bigg|_{\mathsf{K}^\star = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}} = \frac{1}{2(1-a)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

Indefinite with eigenvalues:

$0 \text{ and } \pm\frac{1}{2(1-a)}$

Structure of Stationary Points

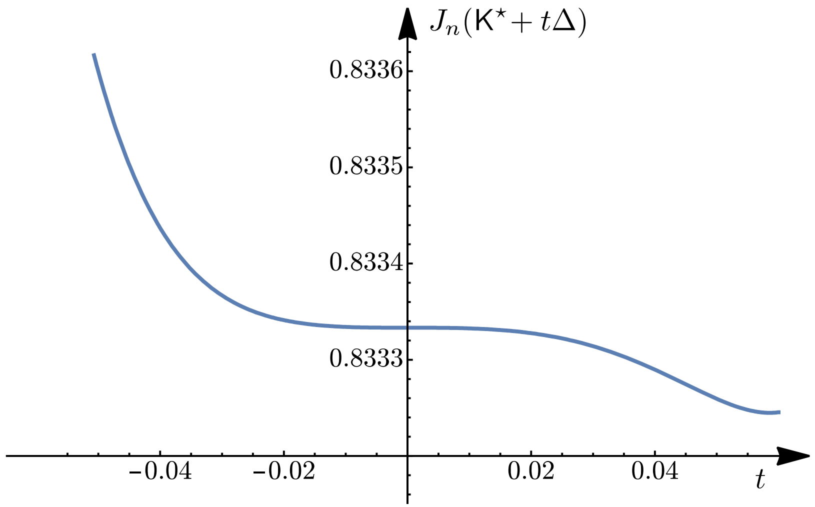
❑ Main Result: existences of strict saddle points

Theorem 4: Consider any open-loop stable plant. The zero controller with any stable A_{K}

$$\mathsf{K} = (A_{\mathsf{K}}, 0, 0) \in \mathcal{C}_{\text{full}}$$

is a stationary point. Furthermore, the corresponding hessian is either indefinite (strict saddle point) or equal to zero.

Another example with zero Hessian



How does the set of Stationary Points look like?

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \begin{aligned} \frac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} &= 0, \\ \frac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} &= 0, \\ \frac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} &= 0, \end{aligned} \right\}$$

- ❑ **Non-unique, non-isolated**
- ❑ **Strictly suboptimal points; Strict saddle points**
- ❑ **All bad stationary points correspond to non-minimal controllers**

Structure of Stationary Points

❑ Main Result

Theorem 5: **All stationary points corresponding to controllable and observable controllers are globally optimum.**

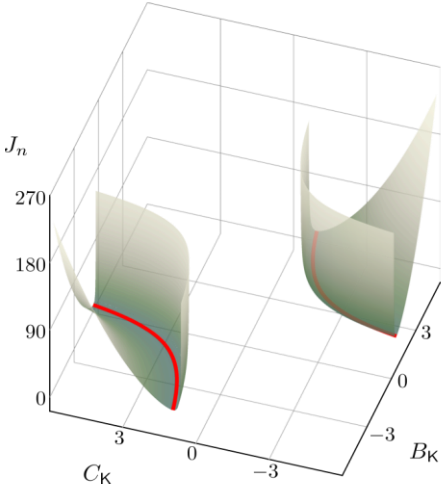
$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \begin{array}{l} \dfrac{\partial J(\mathsf{K})}{\partial A_{\mathsf{K}}} = 0, \\ \dfrac{\partial J(\mathsf{K})}{\partial B_{\mathsf{K}}} = 0, \\ \dfrac{\partial J(\mathsf{K})}{\partial C_{\mathsf{K}}} = 0, \end{array} \right\}$$

Particularly, given a stationary point that is a **minimal** controller

1) This stationary point is a global optimum of $J(\mathsf{K})$
2) The set of all global optima forms a manifold with 2 connected components. They are connected by a similarity transformation.

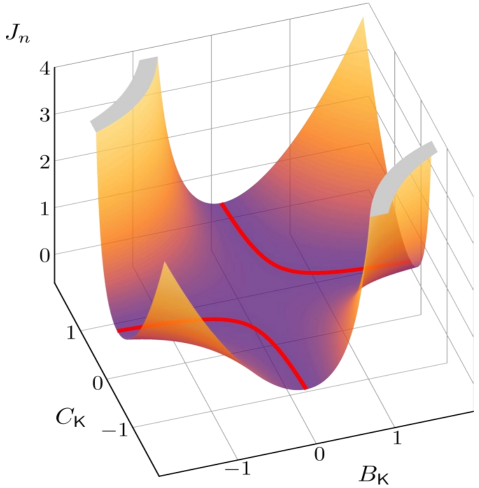
Example: open-loop unstable system

$$\dot{x}(t) = x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$



Example: open-loop stable system

$$\dot{x}(t) = -x(t) + u(t) + w(t)$$
$$y(t) = x(t) + v(t)$$



Proof idea

❑ Proof: all minimal stationary points are unique up to a similarity transformation

All minimal stationary points $\mathsf{K} = (A_\mathsf{K}, B_\mathsf{K}, C_\mathsf{K}) \in \mathcal{C}_{\text{full}}$ to the LQG problem are in the form of

$$A_\mathsf{K} = T(A - BK - LC)T^{-1}, \qquad B_\mathsf{K} = -TL, \qquad C_\mathsf{K} = KT^{-1},$$

$$K = R^{-1}B^\mathsf{T}S, \; L = PC^\mathsf{T}V^{-1},$$

T is an invertible matrix and P, S are the unique positive definite solutions to the Riccati equations

$$\left\{ \mathsf{K} \in \mathcal{C}_{\text{full}} \;\middle|\; \frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0, \; \frac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 0, \; \frac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 0, \right\}$$

Minimal controller

$$\frac{\partial J_n(\mathsf{K})}{\partial B_\mathsf{K}} = 0 \quad \Longrightarrow \quad B_\mathsf{K} = -TPC^\mathsf{T}V^{-1}$$

$$\frac{\partial J_n(\mathsf{K})}{\partial C_\mathsf{K}} = 0 \quad \Longrightarrow \quad C_\mathsf{K} = R^{-1}B^\mathsf{T}ST^{-1}$$

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 0 \quad \Longrightarrow \quad A_\mathsf{K} = T(A - PC^\mathsf{T}V^{-1}C - BR^{-1}B^\mathsf{T}S)T^{-1}$$

$$\frac{\partial J(\mathsf{K})}{\partial A_\mathsf{K}} = 2\left(Y_{12}^\mathsf{T}X_{12} + Y_{22}X_{22}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial B_\mathsf{K}} = 2\left(Y_{22}B_\mathsf{K}V + Y_{22}X_{12}^\mathsf{T}C^\mathsf{T} + Y_{12}^\mathsf{T}X_{11}C^\mathsf{T}\right),$$

$$\frac{\partial J(\mathsf{K})}{\partial C_\mathsf{K}} = 2\left(RC_\mathsf{K}X_{22} + B^\mathsf{T}Y_{11}X_{12} + B^\mathsf{T}Y_{12}X_{22}\right),$$

Special case in Theorem 20.6 of Zhou et al., 1996 and Section II of Hyland, 1984

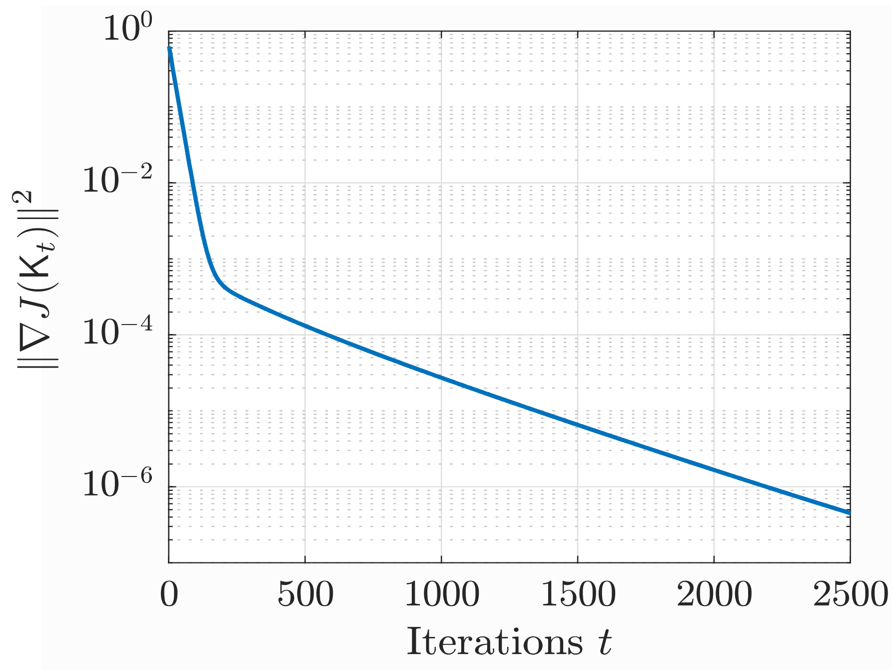
Structure of Stationary Points

❑ Implication

Corollary: Consider gradient descent iterations

$$\mathsf{K}_{t+1} = \mathsf{K}_t - \alpha \nabla J(\mathsf{K}_t)$$

If the iterates converge to a minimal controller, then this minimal controller is a global optima.



Open questions:

- ✓ Convergence conditions?
- ✓ Convergence speed?
- ✓ Alternative model-free parameterization

Some recent papers are

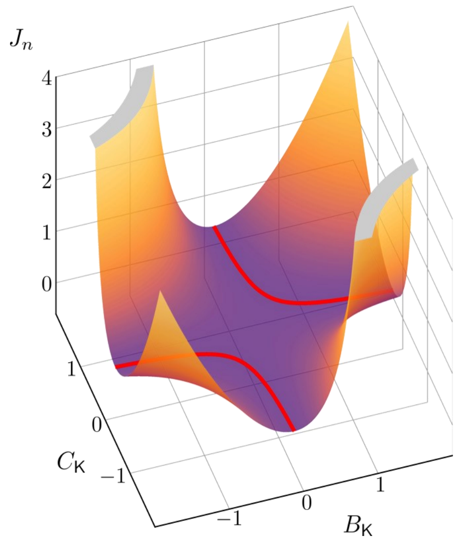
- Umenberger, J., et al. (2022). Globally Convergent Policy Search over Dynamic Filters for Output Estimation. *arXiv preprint arXiv:2202.11659*.
- Zheng, Y., Sun, Y., Fazel, M., & Li, N. (2022). Escaping High-order Saddles in Policy Optimization for Linear Quadratic Gaussian (LQG) Control. arXiv preprint arXiv:2204.00912.

Comparison with LQR

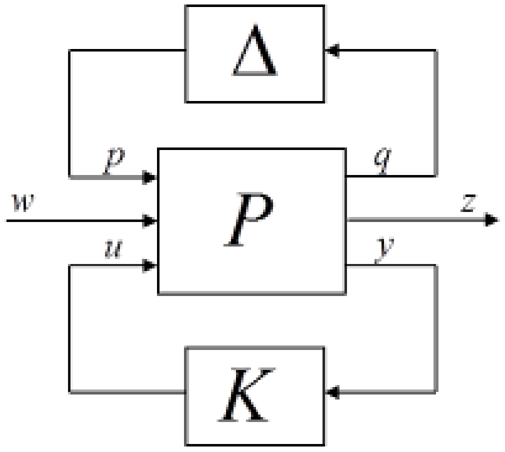
	LQR as an Optimization problem	LQG as an Optimization problem
	$\min_{K} \; J(K)$ $\text{s.t.} \; K \in \mathcal{K}$	$\min_{\mathsf{K}} \; J(\mathsf{K})$ $\text{s.t.} \; \mathsf{K} = (A_{\mathsf{K}}, B_{\mathsf{K}}, C_{\mathsf{K}}) \in \mathcal{C}_{\text{full}}$
Connectivity of feasible region	❖ Always connected	❖ Disconnected, but at most 2 connected comp. ❖ They are almost identical to each other
Stationary points	❖ Unique	❖ Non-unique, non-isolated stationary points ❖ Spurious stationary points (strict saddle, nonminimal controller) ❖ **All mini. stationary points are globally optimal**
Gradient Descent	❖ Gradient dominance ❖ Global fast convergence (like strictly convex)	❖ No gradient dominance ❖ Local convergence/speed (**unknown**) ❖ **Many open questions**
References	Fazel et al., ICML, 2018; Malik et al., 2019; Mohammadi et al., IEEE TAC, 2020; Li et al., 2019; K. Zhang, B. Hu, and T. Başar, 2021; Furieri et al., 2019; Feiran Zhao & Keyou You, 2021, and many others	Zheng*, Tang*, Li. 2021, link (* equal contribution)

Today’s talk

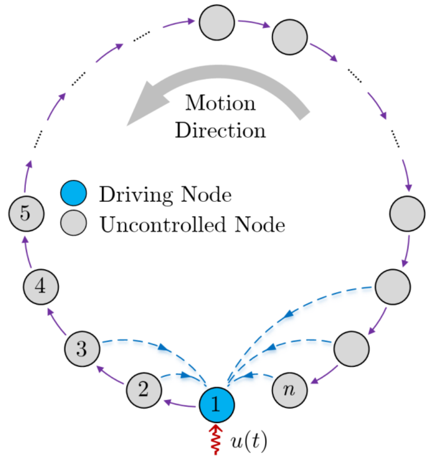
Part 1: Model free LQG control



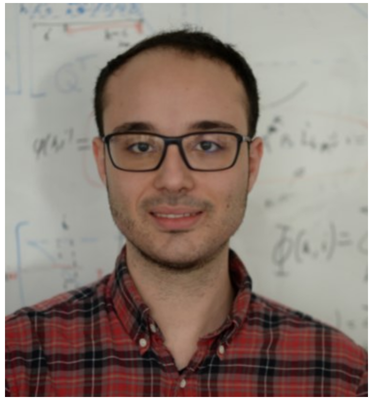
Part 2: Model-based LQG control



Part 3: Data-driven MPC in mixed traffic



Sample complexity of linear quadratic gaussian (LQG) control for output feedback systems



Luca Furieri
EPFL

Maryam Kamgarpour
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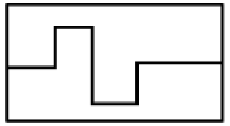
Harvard University

Zheng, Y., Furieri, L., Kamgarpour, M., & Li, N. (2021, May). Sample complexity of linear quadratic gaussian (LQG) control for output feedback systems. In Learning for Dynamics and Control (pp. 559-570). PMLR.

System ID + Robust Control

❑ System ID procedure

Physical systems



Control inputs u_t

Output meas. y_t

Estimate a dynamical model

❑ How to represent a dynamical system: space-space or frequency domain?

- ✓ State-feedback LQR seems easier

$$\hat{A} + \Delta A, \quad \hat{B} + \Delta B, \qquad \|\Delta A\| \leq \epsilon_A, \|\Delta B\| \leq \epsilon_B,$$

- ✓ System-level parameterization (SLP, frequency domain technique) for robust control and sample complexity analysis; see Dean et al., 2020

❑ Partially observed LQG case

Natural idea: estimate $\|\hat{A} - A_\star\|, \quad \|\hat{B} - B_\star\|, \quad \|\hat{C} - C_\star\|,$

Then, design a robust LQG controller?

Highly Non-trivial

- ✓ Dean et al. 2020 works only for state feedback via SLP
- ✓ The realization of A, B, C is not unique!!

Frequency domain formulation

❑ State-space model

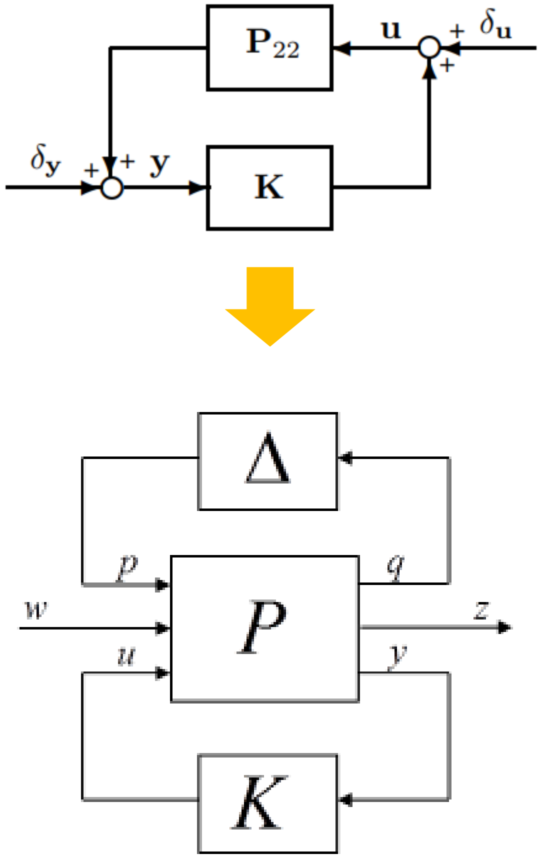
$$x_{t+1} = A_\star x_t + B_\star u_t + B_\star w_t,$$
$$y_t = C_\star x_t + v_t\,.$$

❑ Unique transfer function

$$\mathbf{G}_\star(z) = C_\star(zI - A_\star)^{-1}B_\star\,,$$

Estimate a nominal model
as well as its uncertainty

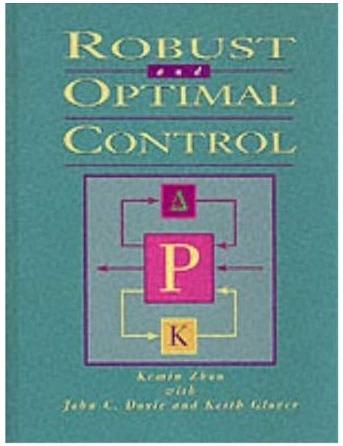
$$\|\boldsymbol{\Delta}\|_\infty := \|\mathbf{G}_\star - \hat{\mathbf{G}}\|_\infty < \epsilon$$

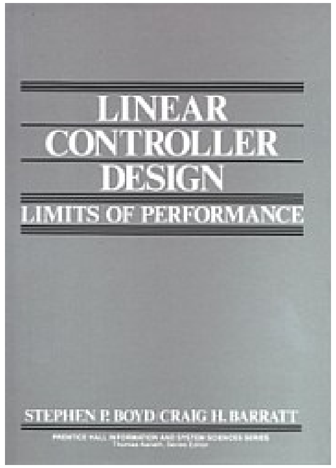


Least-square fits a coarse model

High dimen. stats bounds the error

Design a robust LQG controller





Robust LQG formulation

Nominal LQG formulation

$$\min_{u_0, u_1, \ldots} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^{\mathsf{T}} Q y_t + u_t^{\mathsf{T}} R u_t\right)\right]$$

$$\text{subject to} \quad x_{t+1} = A_\star x_t + B_\star u_t + B_\star w_t,$$
$$y_t = C_\star x_t + v_t \, . \, .$$

Robust LQG formulation

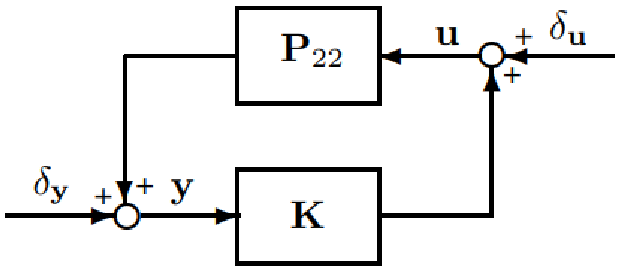
$$\min_{\mathbf{K}} \sup_{\|\mathbf{\Delta}\|_\infty < \epsilon} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^{\mathsf{T}} Q y_t + u_t^{\mathsf{T}} R u_t\right)\right],$$

$$\text{subject to} \quad \mathbf{y} = (\hat{\mathbf{G}} + \mathbf{\Delta})\mathbf{u} + \mathbf{v}$$
$$\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{w},$$

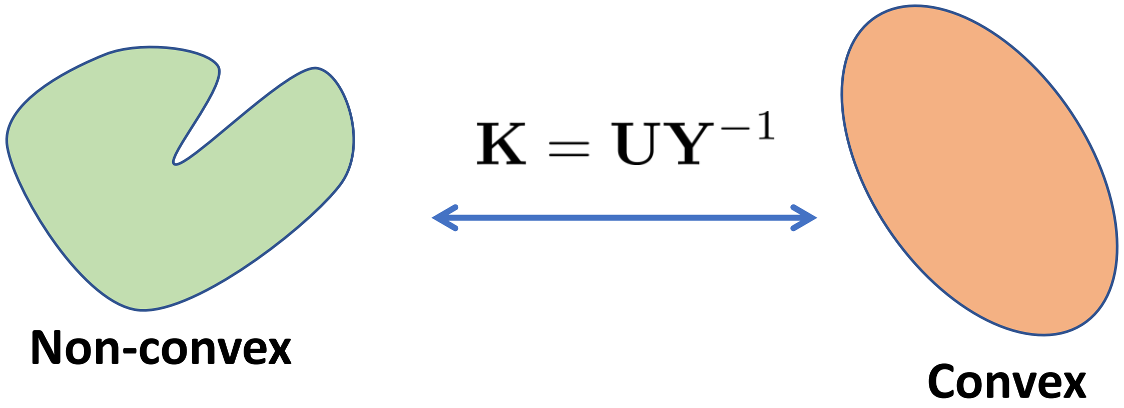
Key idea via Change of variables:

Instead of optimizing the controller K,
we search over the closed-loop responses

Closed-loop convexity



$$\begin{bmatrix}\mathbf{y}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{Y} & \mathbf{W}\\ \mathbf{U} & \mathbf{Z}\end{bmatrix}\begin{bmatrix}\delta_y\\ \delta_u\end{bmatrix}$$



$$(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z}) \in \hat{\mathcal{C}}$$
$$\hat{\mathcal{C}} \equiv \text{Affine space}$$
$$\cap \ \text{Stable}$$

Furieri, L., Zheng, Y., Papachristodoulou, A., & Kamgarpour, M. (2019). An input–output parametrization of stabilizing controllers: Amidst youla and system level synthesis. *IEEE Control Systems Letters*, *3*(4), 1014-1019.

Robust LQG formulation

Robust LQG formulation

$$\min_{\mathbf{K}} \sup_{\|\mathbf{\Delta}\|_\infty<\epsilon} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^{\mathsf{T}}Qy_t + u_t^{\mathsf{T}}Ru_t\right)\right],$$

$$\text{subject to} \quad \mathbf{y} = (\hat{\mathbf{G}}+\mathbf{\Delta})\mathbf{u}+\mathbf{v}$$

$$\mathbf{u} = \mathbf{K}\mathbf{y}+\mathbf{w},$$

Theorem (Zheng et al., 2021): the problem above is equivalent to

$$\min_{\hat{\mathbf{Y}},\hat{\mathbf{W}},\hat{\mathbf{U}},\hat{\mathbf{Z}}} \sup_{\|\mathbf{\Delta}\|_\infty<\epsilon} J(\mathbf{G}_\star,\mathbf{K}) = \left\|\begin{bmatrix}\hat{\mathbf{Y}}(I-\mathbf{\Delta}\hat{\mathbf{U}})^{-1} & \hat{\mathbf{Y}}(I-\mathbf{\Delta}\hat{\mathbf{U}})^{-1}(\hat{\mathbf{G}}+\mathbf{\Delta}) \\ \hat{\mathbf{U}}(I-\mathbf{\Delta}\hat{\mathbf{U}})^{-1} & (I-\hat{\mathbf{U}}\mathbf{\Delta})^{-1}\hat{\mathbf{Z}}\end{bmatrix}\right\|_{\mathcal{H}_2}$$

$$\text{subject to} \quad \begin{bmatrix}I & -\hat{\mathbf{G}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix},$$

$$\begin{bmatrix}\hat{\mathbf{Y}} & \hat{\mathbf{W}} \\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix}\begin{bmatrix}-\hat{\mathbf{G}} \\ I\end{bmatrix} = \begin{bmatrix}0 \\ I\end{bmatrix},$$

$$\hat{\mathbf{Y}},\hat{\mathbf{W}},\hat{\mathbf{U}},\hat{\mathbf{Z}} \in \mathcal{RH}_\infty, \ \|\hat{\mathbf{U}}\|_\infty \leq \frac{1}{\epsilon},$$

Another upper approximation via Taylor expansion

→ Convex optimization

Suboptimality guarantee

Theorem (Zheng et al., 2021): When the plant is open-loop stable, solving an SDP upper approximation of the robust control problem leads to a robust stabilizing LQG control with a suboptimality gap

$$\frac{J(\hat{\mathbf{K}}) - J_\star}{J_\star} \leq 20\epsilon\|\mathbf{U}_\star\|_\infty + \mathcal{O}(\epsilon)\,,$$

where $\|\mathbf{G}_\star - \hat{\mathbf{G}}\|_\infty < \epsilon$, and the estimation is accurate enough

Optimality vs. Robustness

- Certainty equivalent controller (Mania et al., 2019) achieves a better sub-optimality scaling $\mathcal{O}(\epsilon^2)$
- Much stricter requirement on admissible uncertainty,
- No guarantee of robust stabilization performance

"The price of obtaining a faster rate for LQR is that the certainty equivalent controller becomes less robust to model uncertainty"

End-to-end Sample complexity

Nominal LQG formulation

$$\min_{u_0,u_1,\dots} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^{\mathsf{T}} Q y_t + u_t^{\mathsf{T}} R u_t\right)\right]$$
$$\text{subject to} \quad x_{t+1} = A_\star x_t + B_\star u_t + B_\star w_t,$$
$$y_t = C_\star x_t + v_t\,. .$$

Robust LQG formulation

$$\min_{\mathbf{K}} \sup_{\|\mathbf{\Delta}\|_\infty < \epsilon} \quad \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^{\mathsf{T}} Q y_t + u_t^{\mathsf{T}} R u_t\right)\right],$$
$$\text{subject to} \quad \mathbf{y} = (\hat{\mathbf{G}} + \mathbf{\Delta})\mathbf{u} + \mathbf{v}$$
$$\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{w},$$

End-to-end Sample complexity:

Suppose the true plant is FIR of order T_0 and let the length $T \geq T_0$. With high probability, the end-to-end sample complexity scales as

$$\frac{J(\hat{\mathbf{K}}) - J_\star}{J_\star} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$$

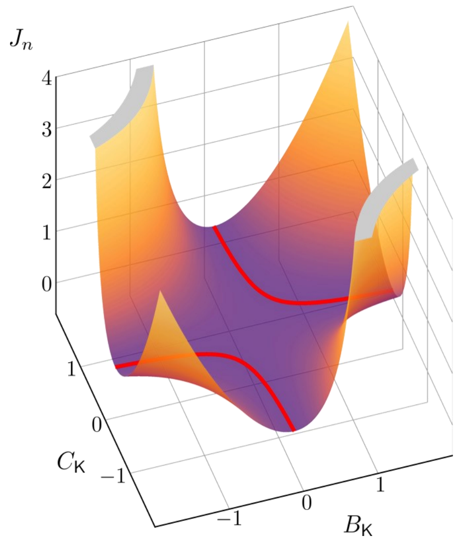
- N is the number of samples (y_t, u_t) in a single trajectory
- **Robust stability**: as long as the Robust LQG has a feasible solution, the closed-loop is guaranteed to be stable:

Comparison with LQR

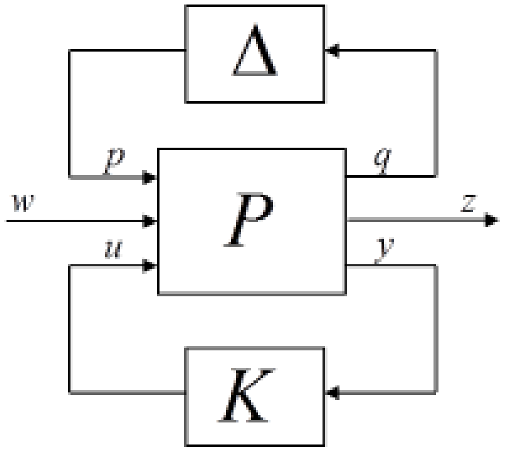
	LQR	Input-output approach
	$\min_{\mathbf{K}} \sup_{\\|\Delta_A\\|,\\|\Delta_B\\|<\epsilon} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}(x_t^\mathsf{T} Q x_t + u_t^\mathsf{T} R u_t)\right]$ subject to $x_{t+1} = (\hat{A}+\Delta A)x_t + (\hat{B}+\Delta B)u_t + v_t$, $\mathbf{u} = \mathbf{K}\mathbf{x}$	$\min_{\mathbf{K}} \sup_{\\|\boldsymbol{\Delta}\\|_\infty<\epsilon} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t)\right]$ subject to $\mathbf{y} = (\hat{\mathbf{G}}+\boldsymbol{\Delta})\mathbf{u} + \mathbf{v}$, $\mathbf{u} = \mathbf{K}\mathbf{y} + \mathbf{w}$,
Sys ID methods	Least squares: $\\|\hat{A}-A_\star\\| \le \epsilon_A, \\|\hat{B}-B_\star\\| \le \epsilon_B,$	Least squares: $\\|\boldsymbol{\Delta}\\|_\infty := \\|\mathbf{G}_\star - \hat{\mathbf{G}}\\|_\infty < \epsilon$
Synthesis Technique	Frequency domain; System-level synthesis, SLS (Wang et al., 2019); Taylor expansion	Frequency domain; Input-output parameterization, IOP, (Furieri et al., 2019); Taylor expansion
Sample Complexity	both stable and unstable systems: $\frac{J(\hat{K})-J_\star}{J_\star} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right),$	Only for open-loop stable system: $\frac{J(\hat{\mathbf{K}})-J_\star}{J_\star} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right);$
References	✓ Dean et al., 2020; Berberich et al., 2020; Boczar et al., 2018; Tsiamis et al., 2020; Umenberger et al., 2019; and many others	• Zheng*, Furieri*, Kamgarpour, & Li, (2021, May). Link (equal contribution)

Today's talk

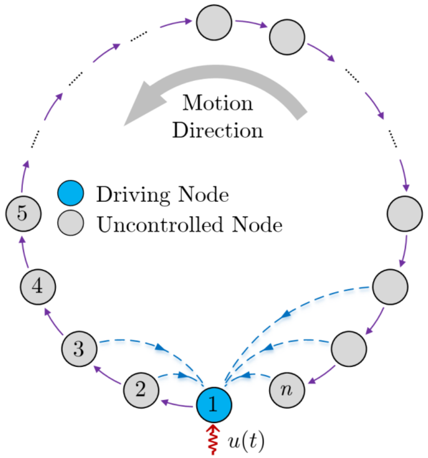
Part 1: Model free LQG control



Part 2: Model-based LQG control



Part 3: Data-driven MPC in mixed traffic



Data-Driven Predictive Control for Connected and Autonomous Vehicles in Mixed Traffic

Jiawei Wang
Tsinghua University

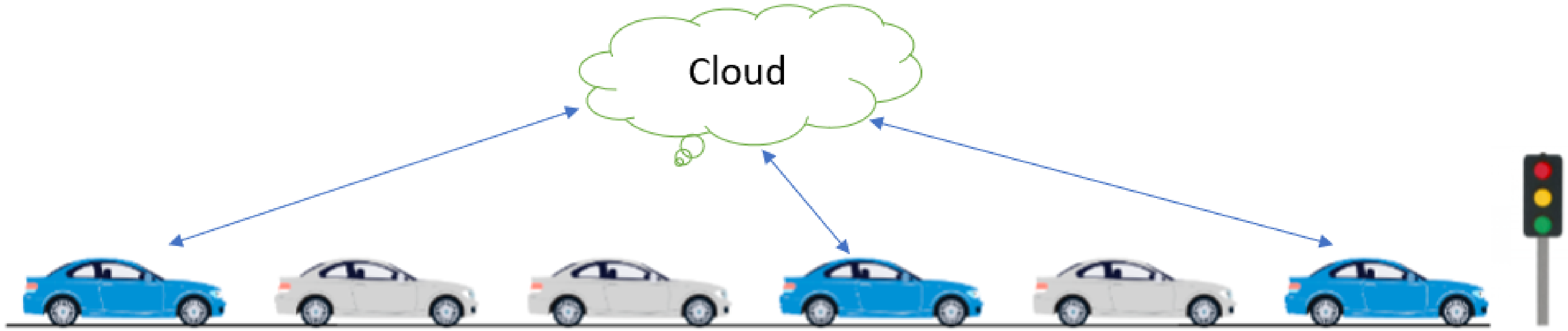
Qing Xu
Tsinghua University

Keqiang Li
Tsinghua University

Wang, J., Zheng, Y., Li, K., & Xu, Q. (2022). DeeP-LCC: Data-enabled predictive leading cruise control in mixed traffic flow. arXiv preprint arXiv:2203.10639.

Mix-Autonomy Mobility

❑ A long stage of mixed-autonomy mobility



Mixed-autonomy mobility: a traffic condition where both autonomous vehicles and human-driven vehicles co-exist.

- **Q1:** How will **a small scale of autonomous vehicles** change traffic dynamics?
- **Q2:** How to integrate **a small scale of autonomous vehicles** to improve traffic performance?

Benchmark Ring Road Experiment

Traffic jams

Sugiyama, et al.

1950s — >10,000 papers for traffic control — **2008** — 2022

Setting:

22 human drivers

Instructions:

drive at 30 km/h /following its preceding vehicle

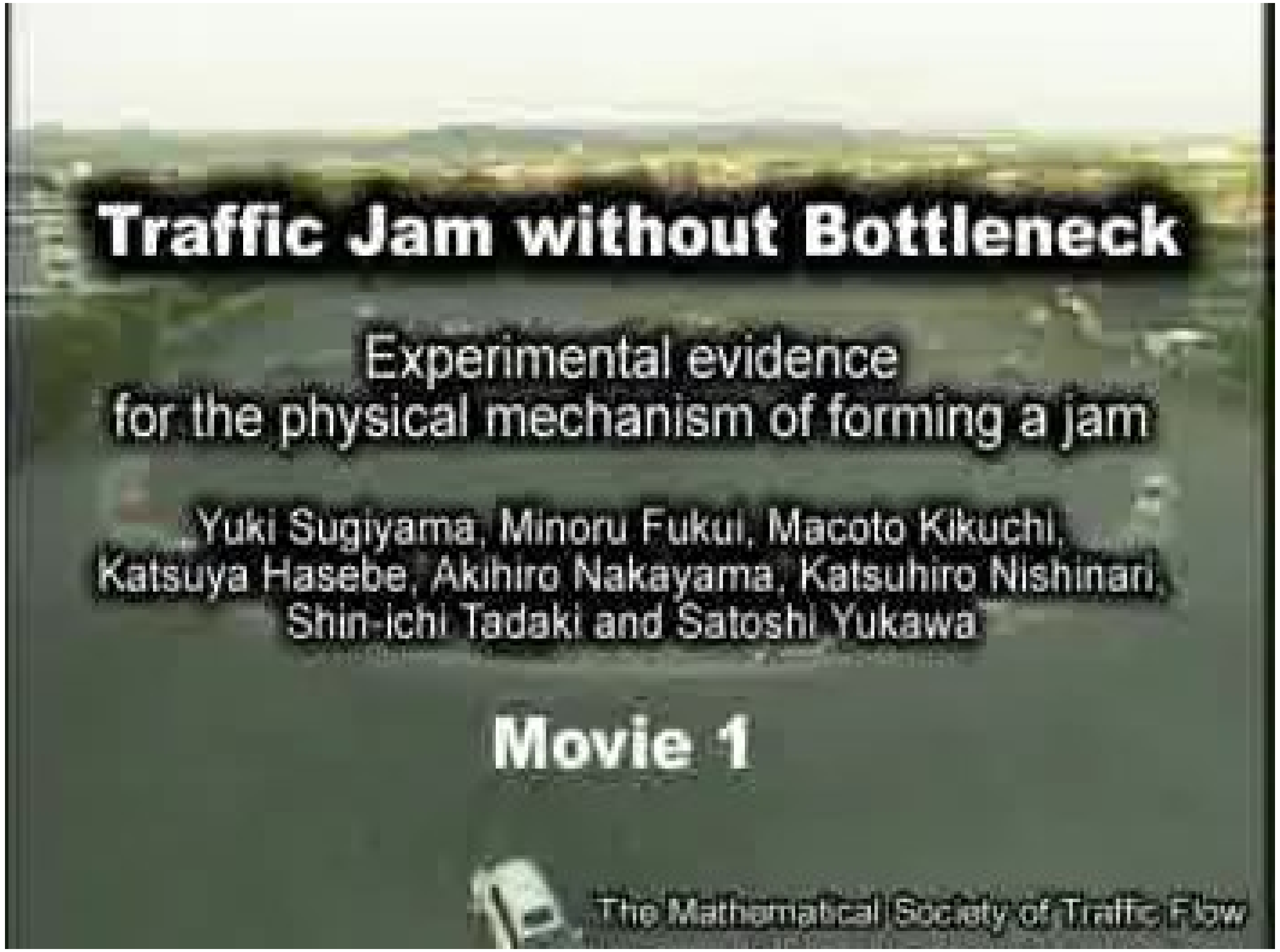
Environment

Single lane

No traffic lights,

No stop signs,

No lane changes.



Video credits: NewScientist.com

Benchmark Ring Road Experiment

Traffic jams

Setting:

21 human drivers

+ 1 AV

Instructions:

drive at 30km/h /follow its preceding vehicle

Environment

Single lane

No traffic lights,

No stop signs,

No lane changes.

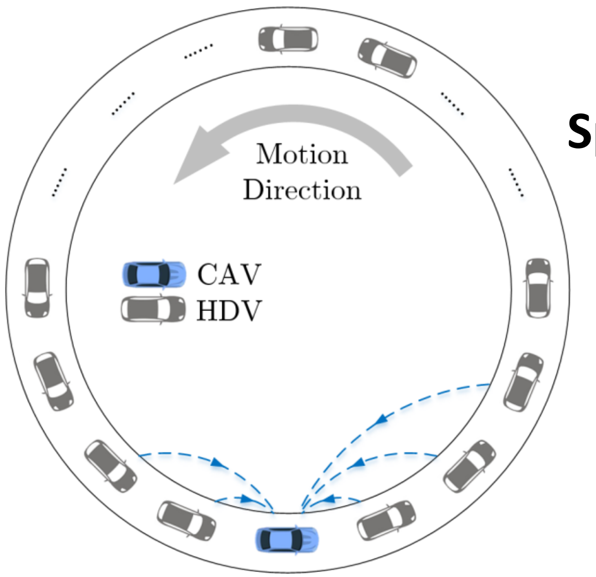
Dissipation of stop-and-go traffic waves via control of a single autonomous vehicle

ILLINOIS UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN · RUTGERS · TEMPLE UNIVERSITY · THE UNIVERSITY OF ARIZONA

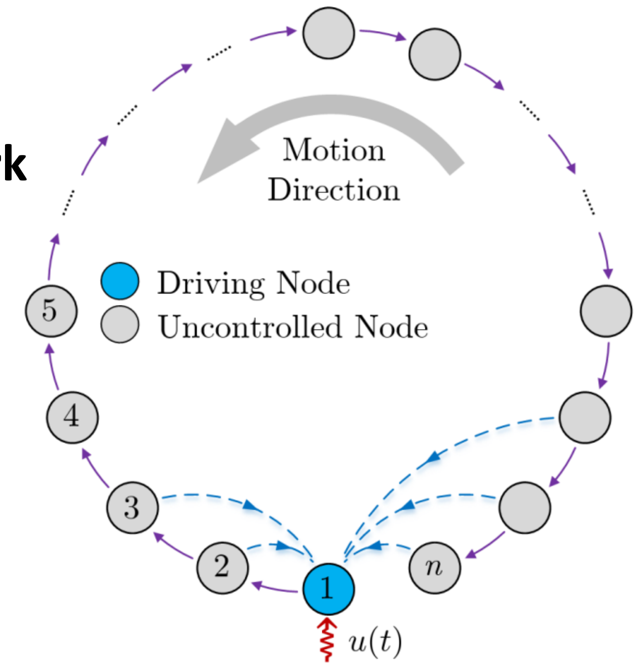
Mixed urban mobility

❑ Theoretical Evidence & Controller design

- Why does it work?
- Does it work in other setups (e.g., different number of HDVs, different human-driver behavior, open straight road scenario)?



Sparse network control



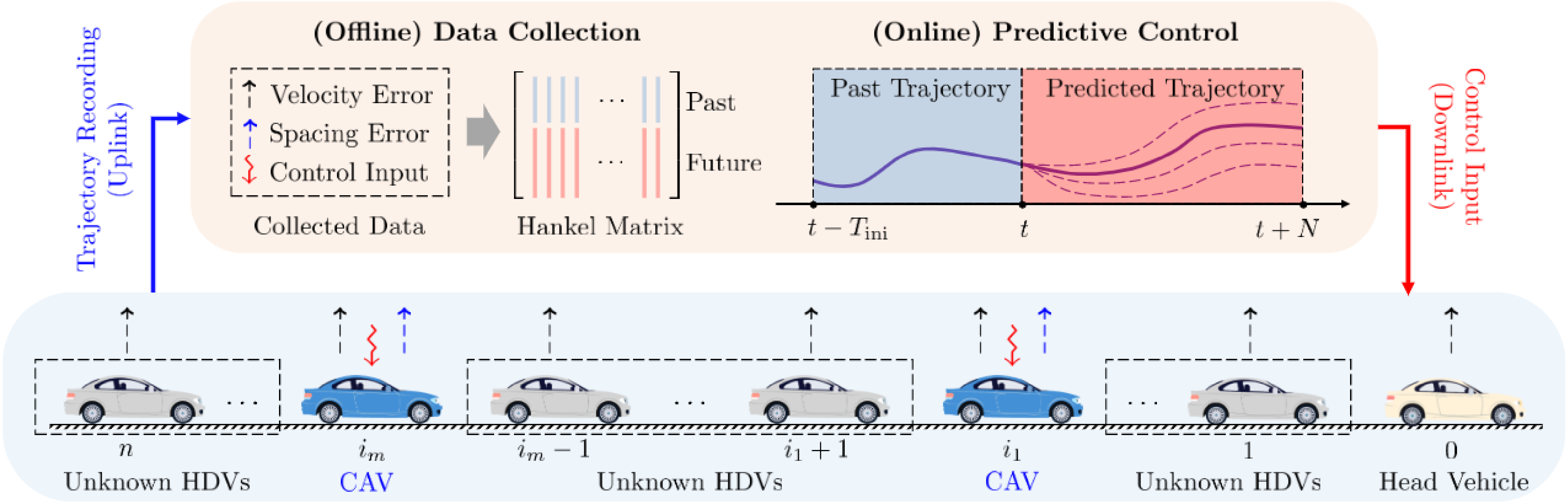
- ❑ **Theorem (Informal):** The mixed traffic system is **stabilizable** after introducing a single autonomous vehicle;
- ❑ Design a distributed controller;

$$\begin{aligned} \text{minimize} \quad & J(K) \\ \text{subject to} \quad & K \in \mathcal{C} \cap \text{Sparse}(S). \end{aligned}$$

- Zheng, Y., Wang, J., & Li, K. (2020). Smoothing traffic flow via control of autonomous vehicles. IEEE Internet of Things Journal, 7(5), 3882-3896.
- Wang, J., Zheng, Y., Xu, Q., Wang, J., & Li, K. (2020). Controllability analysis and optimal control of mixed traffic flow with human-driven and autonomous vehicles. IEEE Transactions on Intelligent Transportation Systems, 22(12), 7445-7459.

Data-driven Leading Cruise Control

❑ System architecture



Wang, J., Zheng, Y., Li, K., & Xu, Q. (2022). DeeP-LCC: Data-enabled predictive leading cruise control in mixed traffic flow. arXiv preprint arXiv:2203.10639.

Data-driven Leading Cruise Control

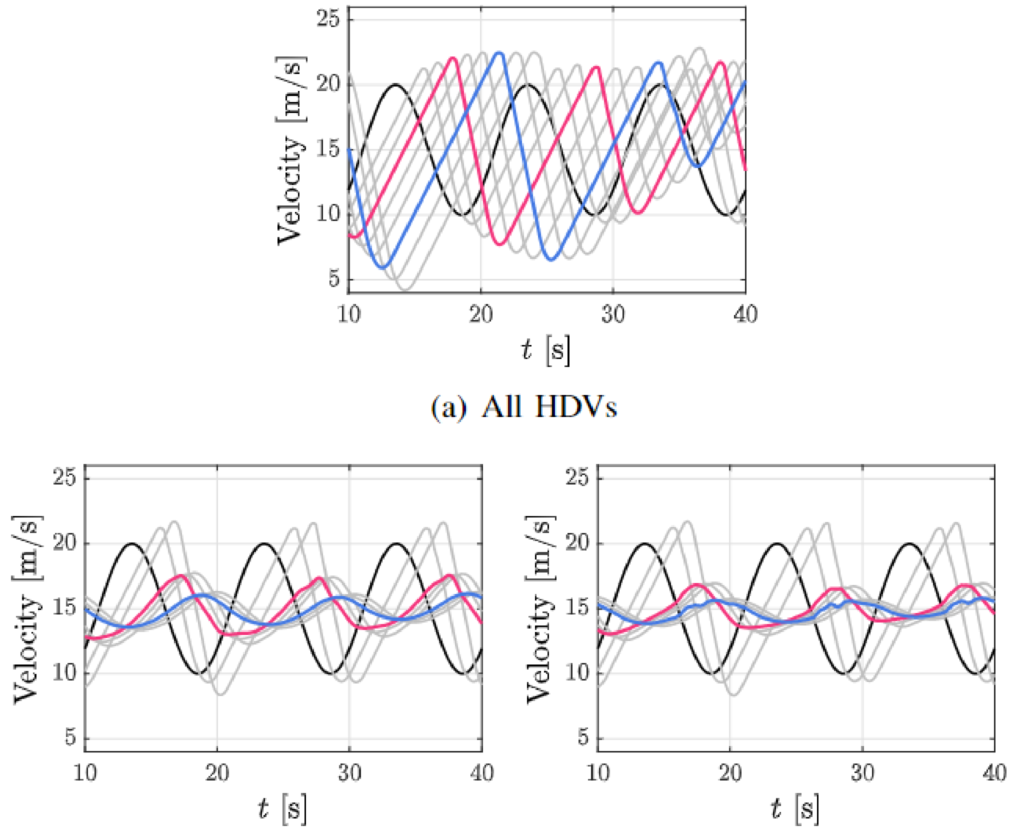
DeeP-LCC: Data-EnablEd Predictive Leading Cruise Control

$$\min_{g,u,y,\sigma_y} \quad \sum_{k=t}^{t+N-1} \left(\|y(k)\|_Q^2 + \|u(k)\|_R^2 \right) + \lambda_g \|g\|_2^2 + \lambda_y \|\sigma_y\|_2^2$$

$$\text{s.t.} \quad \begin{bmatrix} U_\mathrm{p} \\ E_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \\ E_\mathrm{f} \\ Y_\mathrm{f} \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ \epsilon_\mathrm{ini} \\ y_\mathrm{ini} \\ u \\ \epsilon \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sigma_y \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

assumption on ϵ,

safety constraint on u and y.



(a) All HDVs

(b) MPC

(c) DeeP-LCC

Real experiments

❑ Experiment platform

Size: 9 m × 5 m (~500 square ft)

Vehicle: 1.4kg, 0.2m × 0.2m × 0.13m

Operating systems

ubuntu

Ubuntu 18.04.5 LTS

Ubuntu18.04

+

ROS-melodic

@ Tsinghua University https://youtu.be/ZZ2cWhapqpc

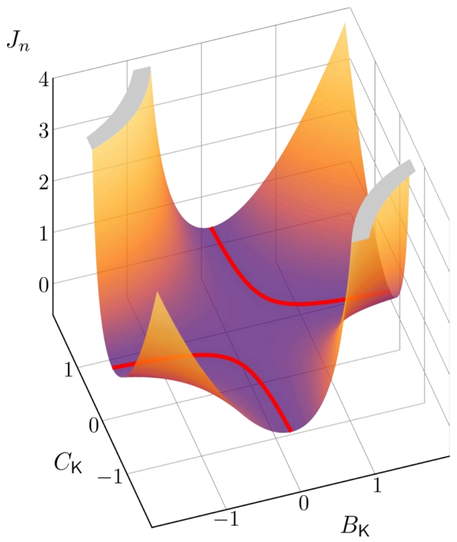
Traffic jams — **Ours**

1950s — >10,000 papers for traffic control — 2022

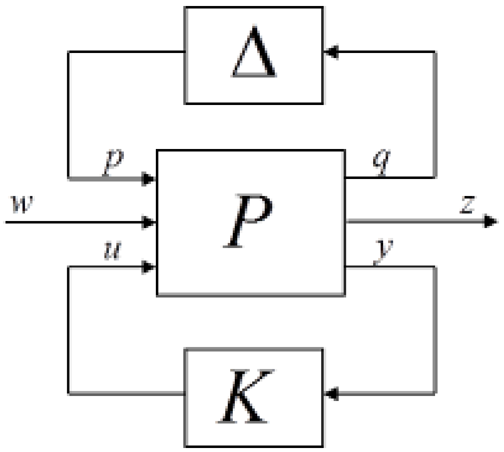
Conclusion

Summary

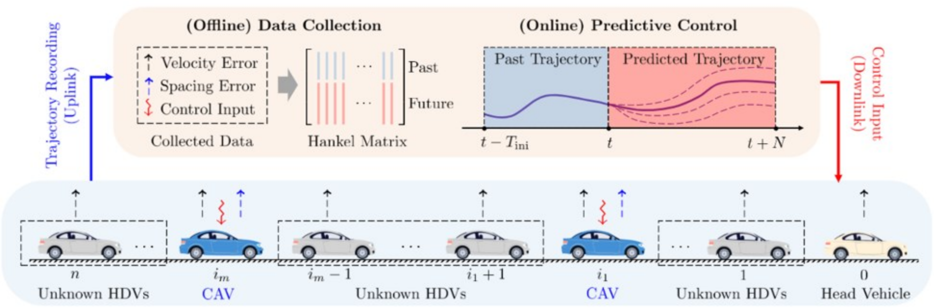
Landscape analysis of non-convex LQG control



Robust Model-based LQG control



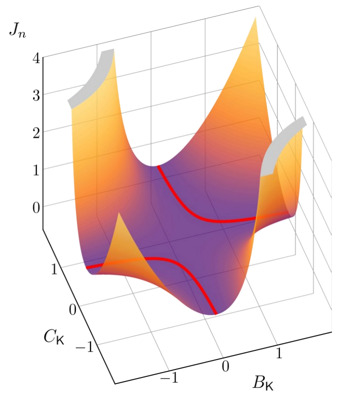
Data-driven MPC in mixed traffic



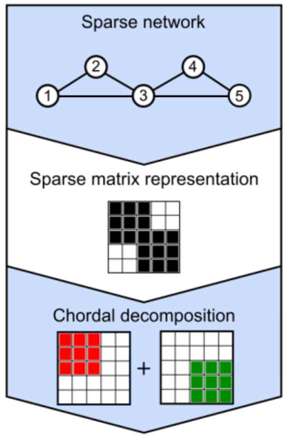
SOC lab at UC San Diego

SOC Lab

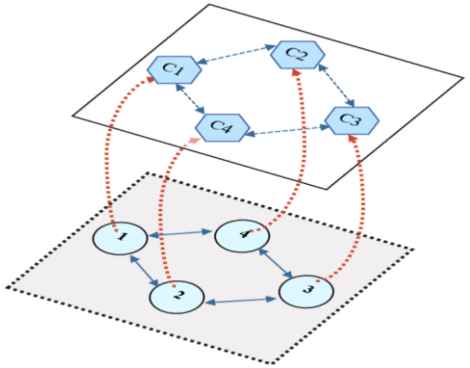
Data-driven and learning-based control



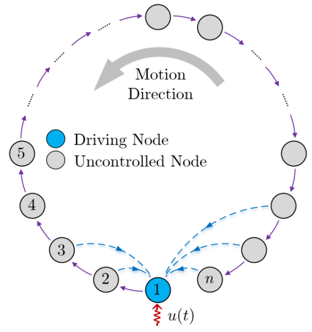
Sparse conic optimization



Scalable distributed control



Connected and autonomous vehicles (CAVs)



Check out our webpage: https://zhengy09.github.io/soclab.html

Scalable Learning, Optimization, and Control for Autonomous Systems

Thank you for your attention!

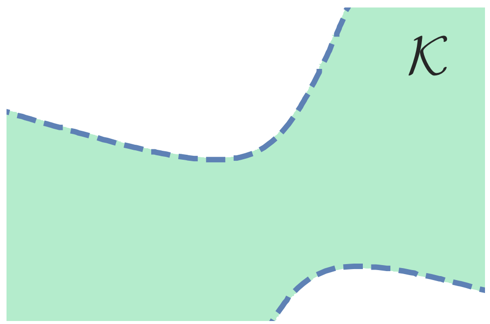
Q & A

Proof idea: Lifting via Change of Variables

❑ Change of variables in state-space domain: Lyapunov theory

- Connectivity of the static stabilizing state feedback gains

$$
\begin{aligned}
&\{K \in \mathbb{R}^{m\times n} \mid A - BK \text{ is stable}\} \\
\iff &\{K \in \mathbb{R}^{m\times n} \mid \exists P \succ 0, P(A-BK)^\mathsf{T} + (A-BK)P \prec 0\} \\
\iff &\{K \in \mathbb{R}^{m\times n} \mid \exists P \succ 0, PA^\mathsf{T} - L^\mathsf{T}B^\mathsf{T} + AP - BL \prec 0, L = KP\} \\
\iff &\{K = LP^{-1} \in \mathbb{R}^{m\times n} \mid \exists P \succ 0, PA^\mathsf{T} - L^\mathsf{T}B^\mathsf{T} + AP - BL \prec 0\}.
\end{aligned}
$$



Open, connected, possibly nonconvex

- How about the set of stabilizing dynamical controllers

$$
\begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix} \text{ is stable}
$$

$$
\iff \quad \exists P \succ 0,\ P\begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix}^\mathsf{T} + \begin{bmatrix} A & BC_\mathsf{K} \\ B_\mathsf{K}C & A_\mathsf{K} \end{bmatrix} P \prec 0,
$$

Change of variables for output feedback control is highly non-trivial

[Gahinet and Apkarian, 1994]
[Scherer et al., IEEE TAC 1997]

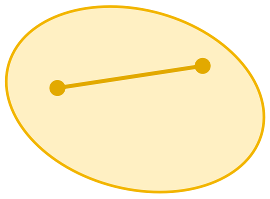
Proof idea: Lifting via Change of Variables

❑ Change of variables in state-space domain: Lyapunov theory

$$\Phi(\mathsf{Z}) = \begin{bmatrix} \Phi_D(\mathsf{Z}) & \Phi_C(\mathsf{Z}) \\ \Phi_B(\mathsf{Z}) & \Phi_A(\mathsf{Z}) \end{bmatrix} := \begin{bmatrix} I & 0 \\ YB & \Xi \end{bmatrix}^{-1} \begin{bmatrix} G & H \\ F & M - YAX \end{bmatrix} \begin{bmatrix} I & CX \\ 0 & \Pi \end{bmatrix}^{-1}.$$

[Scherer et al., IEEE TAC 1997]
[Gahinet and Apkarian, 1994]

$$\mathcal{F} \times \mathrm{GL} \xrightarrow[\text{surjective}]{\Phi} \mathcal{C}_{\text{full}}$$



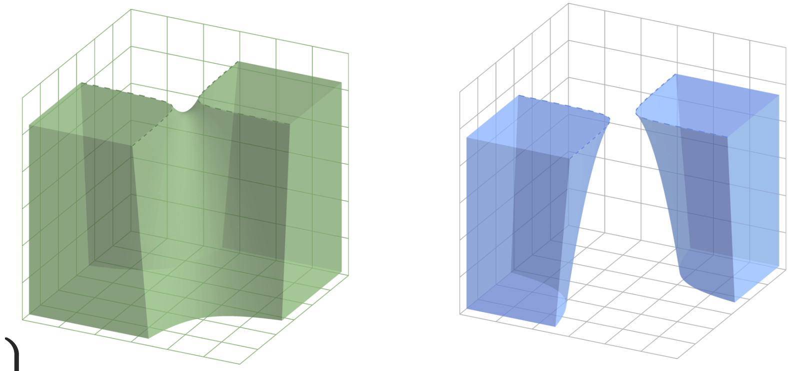
Convex thus connected

General linear group: the set of invertible matrices (similarity transformation)

at most 2 connected components

Two connected components

$$\mathrm{GL}_n^+ = \{\Pi \in \mathbb{R}^{n \times n} \mid \det \Pi > 0\},$$
$$\mathrm{GL}_n^- = \{\Pi \in \mathbb{R}^{n \times n} \mid \det \Pi < 0\}.$$



$$\mathcal{F} = \left\{ (X, Y, M, H, F) | X, Y \in \mathbb{S}^n, M \in \mathbb{R}^{n \times n}, H \in \mathbb{R}^{n \times p}, F \in \mathbb{R}^{m \times n}, \right.$$
$$\left. \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0, \begin{bmatrix} AX + BF & A \\ M & YA + HC \end{bmatrix} + \begin{bmatrix} AX + BF & A \\ M & YA + HC \end{bmatrix}^{\top} \prec 0 \right\}$$