Convex Parameterization of Stabilizing Controllers and Its Application to Distributed Control

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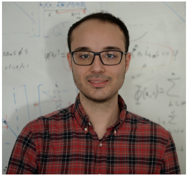
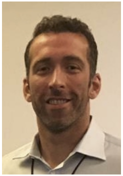
Acknowledgements





Outline

1. Introduction: Stability and non-convexity
2. Closed-loop convexity: Parameterization of stabilizing controllers
3. Explicit equivalence among Youla, SLS, and IOP
4. Other convex parameterizations
5. Conclusion

Introduction: Stability and non-convexity

Stability and non-convexity: static case

Main question: How to represent the set of stabilizing controllers?

- Consider the (centralized) static case with a static state feedback controller $u = Kx$

$$\dot{x} = Ax + Bu$$

- Define the set of stabilizing controllers as

$$\mathcal{C}_1 = \{K \mid A + BK \text{ is stable.}\}$$

- The $\mathcal{C}_1$ is not convex.
- Fortunately, we have a convex representation for the set $\mathcal{C}_1$

$$A + BK \text{ is stable} \Leftrightarrow \exists X \succ 0, (A + BK)X + X(A + BK)^T \prec 0$$

$$\mathcal{C}_{X,Y} = \{X, Y \mid X \succ 0, AX + BY + XA^T + Y^T B^T \prec 0\}$$

- A convex representation of the set of the stabilizing controllers

$$\mathcal{C}_1 = \{YX^{-1} \mid (X, Y) \in \mathcal{C}_{X,Y}\}$$

Stability and non-convexity: dynamic case

Main question: How to represent the set of stabilizing controllers?

A difficult problem: the set of stabilizing distributed controllers

$$\hat{\mathcal{C}}_1 = \{K \in \mathcal{S} \mid A + BK \text{ is stable}\} \qquad \hat{\mathcal{C}}_1 \subseteq \mathcal{C}_1$$

- Even finding one feasible point or verifying whether $\hat{\mathcal{C}}_1$ is empty is nontrivial;

The dynamic case

- Consider the (centralized) case of dynamic controllers $u = \mathbf{K}y$ (detectable and stabilizable)

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\
z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\
y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t).
\end{aligned}$$

- Represent the system in the frequency domain via transfer function matrices.

$$\begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix},$$

where $\mathbf{P}_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$.

Stability and non-convexity: dynamic case

Main question: How to represent the set of stabilizing controllers?

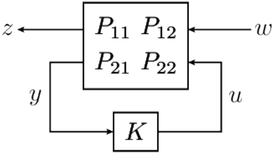


Figure: Linear fractional interconnection of **P** and **K**

- First, define the stability of the closed-loop system.
- Consider a state-space realization of the output feedback controller $u = \mathbf{K}y$

$$\dot{x}_k = A_k x_k + B_k y$$
$$u = C_k x_k + D_k y$$

- **Definition:** the interconnected system is *internally stable* if (x, x_k) is asymptotically stable, *i.e.*, (x, x_k) goes to zero for any initial conditions when $w = 0$.

A state-space condition: the following matrix is stable

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_k \end{bmatrix} \begin{bmatrix} I & -D_k \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_k \\ C & 0 \end{bmatrix}$$

Main question: How to represent the set of stabilizing controllers?

Do we have a condition in frequency domain?

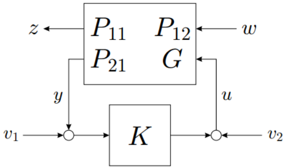


Figure: Linear fractional interconnection of **P** and **K**

A standard notion of stabilization is given as follows:

- **K** stabilizes **G**, if and only if the four transfer matrices from v_1, v_2 to u, y are stable.

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{GK})^{-1} & (I - \mathbf{GK})^{-1}\mathbf{G} \\ \mathbf{K}(I - \mathbf{GK})^{-1} & (I - \mathbf{KG})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}.$$

Stability and non-convexity: dynamic case

Main question: How to represent the set of stabilizing controllers?

- Define a set of stabilizing controllers

$$\mathcal{C}_{\mathbf{G}} = \{\mathbf{K} \text{ internally stabilizes } \mathbf{G}\}.$$

- This set can be equivalently represented by

$$\mathcal{C}_{\mathbf{G}} = \left\{\mathbf{K} \mid \begin{bmatrix} (I-\mathbf{GK})^{-1} & (I-\mathbf{GK})^{-1}\mathbf{G} \\ \mathbf{K}(I-\mathbf{GK})^{-1} & (I-\mathbf{KG})^{-1} \end{bmatrix} \in \mathcal{RH}_\infty \right\}.$$

$$\mathcal{C}_{\mathbf{G}} = \left\{\mathbf{K} \mid \hat{A} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_k \end{bmatrix} \begin{bmatrix} I & -D_k \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_k \\ C & 0 \end{bmatrix} \text{ is stable} \right\}.$$

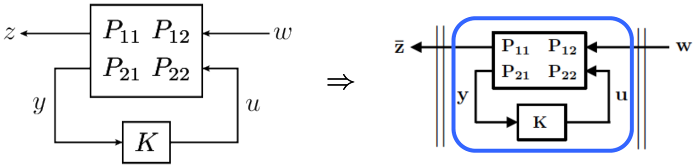
Optimal controller synthesis (including LQR, LQG, $\mathcal{H}_2$, $\mathcal{H}_\infty$ *etc.*)

$$\min_{\mathbf{K}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I-\mathbf{GK})^{-1}\mathbf{P}_{21}\|$$

$$\text{subject to} \quad \mathbf{K} \text{ internally stabilizes } \mathbf{G}.$$

- Both the cost function and feasible region are non-convex in controller $\mathbf{K}$.

Closed-loop convexity: Parameterization of stabilizing controllers



Parameterization: Youla

A classical result to parameterize $\mathcal{C}_{\mathbf{G}}$ is Youla, based on a notion of doubly co-prime factorization.

- Zhou, Kemin, John Comstock Doyle, and Keith Glover. Robust and optimal control. Vol. 40. New Jersey: Prentice hall, 1996.

A collection of stable transfer functions, $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r$ is called a doubly co-prime factorization of $\mathbf{G}$ if

$$\mathbf{G} = \mathbf{N}_r\mathbf{M}_r^{-1} = \mathbf{M}_l^{-1}\mathbf{N}_l$$

and

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = I.$$

- Define the following affine space

$$\mathcal{C}_2 = \{(\mathbf{S}, \mathbf{T}) \mid \mathbf{S} = \mathbf{V}_r - \mathbf{M}_r\mathbf{Q}, \mathbf{T} = \mathbf{U}_r - \mathbf{N}_r\mathbf{Q}, \forall\ \mathbf{Q} \in \mathcal{RH}_\infty\},$$

where $\mathbf{Q}$ is called the **Youla** parameter. It is clear that $\mathcal{C}_2$ is a convex set in $(\mathbf{S}, \mathbf{T})$.

Parameterization: Youla

The set of stabilizing controllers is defined by

$$\begin{aligned}\mathcal{C}_{\mathbf{G}} &= \{\mathbf{K} \text{ stabilizes } \mathbf{G}\} \\ &= \left\{\mathbf{K} \mid \begin{bmatrix}(I-\mathbf{GK})^{-1} & (I-\mathbf{GK})^{-1}\mathbf{G} \\ \mathbf{K}(I-\mathbf{GK}) & (I-\mathbf{KG})\end{bmatrix} \in \mathcal{RH}_\infty\right\}.\end{aligned}$$

Youla Parameterization: it is known that

$$\begin{aligned}\mathcal{C}_{\mathbf{G}} &= \{\mathbf{K} = \mathbf{S}\mathbf{T}^{-1} \mid (\mathbf{S},\mathbf{T}) \in \mathcal{C}_2\} \\ &= \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\}\end{aligned}$$

Optimal controller synthesis

$$\begin{aligned}\min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I-\mathbf{GK})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{G}.\end{aligned}$$

$$\begin{aligned}\min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty,\end{aligned}$$

- It is *an equivalent change of variables* $\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}$ that allows for convexification.

Parameterization: System-level Synthesis (SLS)

A recent result to parameterize $\mathcal{C}_{\mathbf{G}}$ is the so-called *System-level synthesis* (SLS).

- Wang, Y. S., Matni, N., & Doyle, J. C. (2019). A system level approach to controller synthesis. IEEE Transactions on Automatic Control.
- Wang, Y. S., Matni, N., & Doyle, J. C. (2017, May). System level parameterizations, constraints and synthesis. In 2017 American Control Conference (ACC) (pp. 1308-1315). IEEE. (**Best paper award**)

One key observation is still *an equivalent change of variables*, based on the following observations:

- The stability of the following system with controller $u = \mathbf{K}y$

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2u(t) + \delta_x(t),\\ y(t) &= C_2x(t) + \delta_y(t).\end{aligned}$$

  is equivalent to the stability of the following closed-loop transfer functions

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix},$$

Parameterization: System-level Synthesis (SLS)

The closed-loop responses $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ are in the following affine space

$$\begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{1a}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} sI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{1b}$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \frac{1}{s}\mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty. \tag{1c}$$

System-level Parameterization: it is shown that (Wang et al., 2019)

$$\mathcal{C}_{\mathbf{G}} = \{\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \mid \mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L} \text{ are in the affine space (1a)-(1c)}\}$$

Optimal controller synthesis

$$\min_{\mathbf{K}} \quad \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{P}_{21}\|$$

subject to $\mathbf{K}$ internally stabilizes $\mathbf{G}$.

$$\min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{22} \right\|$$

subject to $(1a) - (1c)$.

- It is *an equivalent change of variables* $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$ that allows for convexification.

Parameterization: input-output parameterization (IOP)

Another recent result to parameterize $\mathcal{C}_{\mathbf{G}}$ is the so-called *input-output parameterization*.

- Furieri, L., Zheng, Y., Papachristodoulou, A., & Kamgarpour, M. (2019). An Input-Output Parametrization of Stabilizing Controllers: amidst Youla and System Level Synthesis. IEEE Control Systems Letters.

Our observation is still *an equivalent change of variables*, based on the classical result

- **K** stabilizes **G**, if and only if the four transfer matrices from v_1, v_2 to u, y are stable.

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{GK})^{-1} & (I - \mathbf{GK})^{-1}\mathbf{G} \\ \mathbf{K}(I - \mathbf{GK})^{-1} & (I - \mathbf{KG})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}.$$

Key idea: Treat the closed-loop responses as individual variables that satisfy certain constraints

$$\mathbf{X} = (I - \mathbf{GK})^{-1}$$
$$\mathbf{Y} = \mathbf{K}(I - \mathbf{GK})^{-1}$$
$$\mathbf{W} = (I - \mathbf{GK})^{-1}\mathbf{G}$$
$$\mathbf{Z} = (I - \mathbf{KG})^{-1}$$

Parameterization: input-output parameterization (IOP)

The closed-loop responses $\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}$ are in the following affine space

$$\begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{W} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix}, \tag{2a}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{W} \\ \mathbf{Y} & \mathbf{Z} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{2b}$$

$$\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z} \in \mathcal{RH}_\infty. \tag{2c}$$

Input-output parameterization: the set of stabilizing controllers can be represented as

$$\mathcal{C}_{\mathbf{G}} = \{\mathbf{K} = \mathbf{Y}\mathbf{X}^{-1} \mid \mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z} \text{ are in the affine space (2a)-(2c)}\}.$$

Optimal controller synthesis

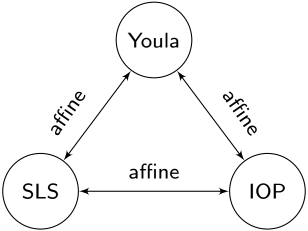
$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{G}. \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{X},\mathbf{Y},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{Y}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (2a) - (2c). \end{aligned}$$

- It is *an equivalent change of variables* $\mathbf{K} = \mathbf{Y}\mathbf{X}^{-1}$ that allows for convexification.

Explicit equivalence among Youla, SLS, and IOP

— any convex SLS can be equivalently reformulated into a convex problem in Youla or IOP; vice versa



Youla $\Leftrightarrow$ IOP

Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{G}$. We have

1. For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\begin{aligned}\mathbf{X} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l\,,\\ \mathbf{Y} &= (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l\,,\\ \mathbf{W} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{N}_l\,,\\ \mathbf{Z} &= I + (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{N}_l\,,\end{aligned}$$

   belong to (2a)-(2c) and are such that $\mathbf{Y}\mathbf{X}^{-1} = (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1}$.

2. For any $(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z})$ in (2a)-(2c), the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l\mathbf{X}\mathbf{U}_r - \mathbf{U}_l\mathbf{Y}\mathbf{U}_r - \mathbf{V}_l\mathbf{W}\mathbf{V}_r + \mathbf{U}_l\mathbf{Z}\mathbf{V}_r - \mathbf{V}_l\mathbf{U}_r\,,$$

   is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and $(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} = \mathbf{Y}\mathbf{X}^{-1}$.

- **Interpretation:** $Ax = b$: $\{x_0 + Av \mid v \text{ is any solution to } Av = 0\}$

$$\begin{bmatrix}\mathbf{X} & \mathbf{W}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix} = \begin{bmatrix}\mathbf{U}_r\mathbf{M}_l & \mathbf{U}_r\mathbf{N}_l\\ \mathbf{V}_l\mathbf{M}_l & I + \mathbf{V}_r\mathbf{N}_l\end{bmatrix} + \begin{bmatrix}\mathbf{N}_r & \\ & \mathbf{M}_r\end{bmatrix}\begin{bmatrix}\mathbf{Q} & \mathbf{Q}\\ \mathbf{Q} & \mathbf{Q}\end{bmatrix}\begin{bmatrix}\mathbf{M}_l & \\ & \mathbf{N}_l\end{bmatrix}$$

IOP $\Leftrightarrow$ SLS

For any $\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}$ satisfying the affine space (1a)-(1c), the transfer matrices

$$\begin{aligned}\mathbf{X} &= C_2\mathbf{N} + I,\\ \mathbf{Y} &= \mathbf{L},\\ \mathbf{W} &= C_2\mathbf{R}B_2,\\ \mathbf{Z} &= \mathbf{M}B_2 + I,\end{aligned}$$

belong to (2a)-(2c) and are such that

$$\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} = \mathbf{Y}\mathbf{X}^{-1}.$$

- The affine relationship can written into

$$\begin{bmatrix}\mathbf{X} & \mathbf{W}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix} = \begin{bmatrix}C_2 & \\ & I\end{bmatrix}\begin{bmatrix}\mathbf{R} & \mathbf{N}\\ \mathbf{M} & \mathbf{L}\end{bmatrix}\begin{bmatrix} & B_2\\ I & \end{bmatrix} + \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}.$$

- *This affine transformation is in general not invertible, but considering the achievability conditions, an explicit converse transformation can be found as well.*

IOP $\Leftrightarrow$ SLS

For any $\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}$ satisfying the affine space (2a)-(2c), the transfer matrices

$$\begin{aligned}
\mathbf{R} &= (sI - A)^{-1} + (sI - A)^{-1} B_2 \mathbf{Y} C_2 (sI - A)^{-1}, \\
\mathbf{M} &= \mathbf{Y} C_2 (sI - A)^{-1}, \\
\mathbf{N} &= (sI - A)^{-1} B_2 \mathbf{Y}, \\
\mathbf{L} &= \mathbf{Y},
\end{aligned}$$

belong to the affine subspace (1a)-(1c) and are such that

$$\mathbf{Y}\mathbf{X}^{-1} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}.$$

Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of $\mathbf{G}$. We have

1. For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices

$$\begin{aligned} \mathbf{R} &= (sI - A)^{-1} + (sI - A)^{-1} B_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l C_2 (sI - A)^{-1} \\ \mathbf{M} &= (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l C_2 (sI - A)^{-1}, \\ \mathbf{N} &= (sI - A)^{-1} B_2 (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \\ \mathbf{L} &= (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q}) \mathbf{M}_l, \end{aligned}$$

   belong to the affine subspace (1a)-(1c) and are such that

$$\mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1}.$$

2. For any $(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L})$ in the affine subspace (1a)-(1c), the transfer matrix

$$\mathbf{Q} = \mathbf{V}_l C_2 \mathbf{N} \mathbf{U}_r - \mathbf{U}_l \mathbf{L} \mathbf{U}_r - \mathbf{V}_l C_2 \mathbf{R} B_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{M} B_2 \mathbf{V}_r + \mathbf{U}_l \mathbf{V}_r$$

   is such that $\mathbf{Q} \in \mathcal{RH}_\infty$ and

$$(\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}.$$

Youla $\Leftrightarrow$ SLS $\Leftrightarrow$ IOP

Convex system-level synthesis: which is claimed to be the largest known class of convex distributed optimal control problems (Wang et al., 2019)

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & g(\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}) \\ \text{subject to} \quad & (1a) - (1c), \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

- This is clearly equivalent to a convex problem in Youla,

$$\begin{aligned} \min_{\mathbf{Q}} \quad & g_1(\mathbf{Q}) \\ \text{subject to} \quad & \begin{bmatrix} f_1(\mathbf{Q}) & f_3(\mathbf{Q}) \\ f_2(\mathbf{Q}) & f_4(\mathbf{Q}) \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

- which is also equivalent to a convex problem in input-output parameterization

$$\begin{aligned} \min_{\mathbf{X},\mathbf{Y},\mathbf{W},\mathbf{Z}} \quad & \hat{g}_1(\mathbf{Y}) \\ \text{subject to} \quad & (2a) - (2c) \\ & \begin{bmatrix} \hat{f}_1(\mathbf{Y}) & \hat{f}_3(\mathbf{Y}) \\ \hat{f}_2(\mathbf{Y}) & \hat{f}_4(\mathbf{Y}) \end{bmatrix} \in \mathcal{S}. \end{aligned}$$

Distributed control

Formulating the problem of distributed control seems to be problem dependent:

- A classical formulation is

$$\begin{aligned} \min_{\mathbf{K}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} \quad & \mathbf{K} \text{ internally stabilizes } \mathbf{G}. \\ & \mathbf{K} \in \mathcal{S} \end{aligned}$$

  which is non-convex in $\mathbf{K}$ no matter what sparsity constraint $\mathcal{S}$ is.

- A recent advertised formulation is the **convex system-level synthesis**

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & g(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}) \\ \text{subject to} \quad & (1a) - (1c), \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \hat{\mathcal{S}}. \end{aligned}$$

  which is convex, as long as $g(\cdot)$ is convex and $\hat{\mathcal{S}}$ is a subspace constraint.

- These two formulations are not directly comparable!
- They can coincide with each other when $\mathcal{S}$ is *quadratic invariant (QI)* w.r.t. $\mathbf{G}$.

Quadratic invariance (QI)

Youla

$$\begin{aligned} \min_{\mathbf{Q}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty, \\ & (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})^{-1} \in \mathcal{S} \end{aligned}$$

$$(\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l \in \mathcal{S}$$

IOP

$$\begin{aligned} \min_{\mathbf{X},\mathbf{Y},\mathbf{W},\mathbf{Z}} \quad & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{Y}\mathbf{P}_{21}\| \\ \text{subject to} \quad & (2a) - (2c). \\ & \mathbf{Y}\mathbf{X}^{-1} \in \mathcal{S} \end{aligned}$$

$$\mathbf{Y} \in \mathcal{S}$$

SLS

$$\begin{aligned} \min_{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right\| \\ \text{subject to} \quad & (1a) - (1c) \\ & \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N} \in \mathcal{S}. \end{aligned}$$

$$\mathbf{L} \in \mathcal{S}$$

If **S** is QI with respect to **G**, the the nonlinear constraint can be equivalently replaced by the linear constraint on the right column.

Other Convex Parameterizations

Convex parameterizations using closed-loop responses

Consider a discrete-time system

$$\begin{aligned} x[t+1] &= Ax[t] + Bu[t] + \delta_x[t], \\ y[t] &= Cx[t] + \delta_y[t], \end{aligned}$$

and a dynamic controller

$$\mathbf{u} = \mathbf{K}\mathbf{y} + \delta_u.$$

- Define the set of stabilizing controllers

$$\mathcal{C}_{\text{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{G}\}.$$

- We can write the closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xx} & \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_u \end{bmatrix},$$

- A classical result $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ if and only if

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \to \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) := \begin{bmatrix} \mathbf{\Phi}_{yy} & \mathbf{\Phi}_{yu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \in \mathcal{RH}_\infty.$$

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When choosing two disturbances and two outputs, we have in total $\binom{3}{2} \times \binom{3}{2} = 9$ choices, *i.e.*,

$$\left(\begin{bmatrix}\delta_x \\ \delta_y\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix}\right), \left(\begin{bmatrix}\delta_x \\ \delta_y\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix}\right), \left(\begin{bmatrix}\delta_x \\ \delta_y\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}\right),$$
$$\left(\begin{bmatrix}\delta_y \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix}\right), \left(\begin{bmatrix}\delta_y \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix}\right), \left(\begin{bmatrix}\delta_y \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}\right),$$
$$\left(\begin{bmatrix}\delta_x \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix}\right), \left(\begin{bmatrix}\delta_x \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix}\right), \left(\begin{bmatrix}\delta_x \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}\right).$$

Under the assumption of stabilizablity and detectablity, we have

- $\mathbf{K}$ internally stabilizes $\mathbf{G}$ if and only if one of the groups of four transfer functions highlighted in black is stable.
- Stability of any other group of 4 closed-loop responses is not sufficient for internal stability.

- $\left(\begin{bmatrix}\delta_y \\ \delta_u\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{y} \\ \mathbf{u}\end{bmatrix}\right) \in \mathcal{RH}_\infty$ is classical and used in IOP; $\left(\begin{bmatrix}\delta_x \\ \delta_y\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{x} \\ \mathbf{u}\end{bmatrix}\right) \in \mathcal{RH}_\infty$ is used in the system-level-synthesis.

Convex parameterizations using closed-loop responses

Case 1:

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$

1. For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses $\mathbf{\Phi}_{yx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy}$ are in the following affine subspace

$$\begin{aligned} \begin{bmatrix} I & -\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} &= \begin{bmatrix} C(zI-A)^{-1} & I \end{bmatrix}, \\ \begin{bmatrix} \mathbf{\Phi}_{yx} & \mathbf{\Phi}_{yy} \\ \mathbf{\Phi}_{ux} & \mathbf{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} zI-A \\ -C \end{bmatrix} &= 0, \\ \mathbf{\Phi}_{yx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy} &\in \mathcal{RH}_\infty. \end{aligned} \tag{3}$$

2. For any transfer matrices $\mathbf{\Phi}_{yx}, \mathbf{\Phi}_{ux}, \mathbf{\Phi}_{yy}, \mathbf{\Phi}_{uy}$ satisfying (3), $\mathbf{K} = \mathbf{\Phi}_{uy}\mathbf{\Phi}_{yy}^{-1} \in \mathcal{C}_{stab}$.

- Case 2 corresponds to the System-level synthesis;
- Case 3 corresponds to the Input-output parameterization.

Convex parameterizations using closed-loop responses

Case 4:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}.$$

1. For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses $\mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xu}, \mathbf{\Phi}_{uu}$ are in the following affine subspace

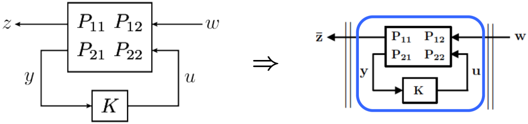
$$\begin{aligned} \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} &= 0 \\ \begin{bmatrix} \mathbf{\Phi}_{xy} & \mathbf{\Phi}_{xu} \\ \mathbf{\Phi}_{uy} & \mathbf{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} &= \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}. \\ \mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xu}, \mathbf{\Phi}_{uu} &\in \mathcal{RH}_\infty, \end{aligned} \tag{4}$$

2. For any transfer matrices $\mathbf{\Phi}_{xy}, \mathbf{\Phi}_{uy}, \mathbf{\Phi}_{xu}, \mathbf{\Phi}_{uu}$ satisfying (4), $\mathbf{K} = \mathbf{\Phi}_{uu}^{-1}\mathbf{\Phi}_{uy} \in \mathcal{C}_{stab}$.

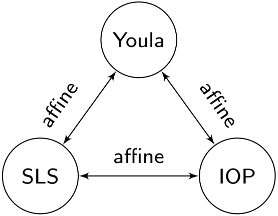
Conclusion

Take-home message

- **Message 1: Closed-loop convexity.** For many controller synthesis problems, one should really consider the convexity in closed-loop form.



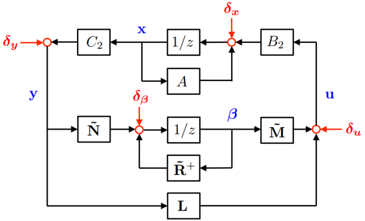
- **Message 2: Youla ⇔ System-level sythesis (SLS) ⇔ Input-output parameterization.** Any convex SLS is also convex in Youla or IOP, and vice versa.



- **Message 3: Distributed Optimal control.** The two formulations are problem dependent, and the existence of QI can make them coincide with each other.

Topics beyond this talk

- **Numerical computation**: even though problems are convex, they are often in infinite dimensional space, and a state-space solution is non-trivial. The FIR approximation in discrete-time is one practical choice.
- **Controller realization and distributed implementation**: Wang et al., 2019



- **Scalable computation**:

  Wang, Y. S., Matni, N., & Doyle, J. C. (2018). Separable and localized system-level synthesis for large-scale systems. IEEE Transactions on Automatic Control, 63(12), 4234-4249.
- **Robust versions and their applications in learning-based control**

  Dean, S., Mania, H., Matni, N., Recht, B., & Tu, S. (2017). On the sample complexity of the linear quadratic regulator. arXiv preprint arXiv:1710.01688.

Thank you for your attention!

Q & A

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