Smoothing Traffic Flow via Control of Autonomous Vehicles

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Outline

1. The potential of autonomous vehicles on traffic dynamics
2. Control-theoretic modeling and analysis
3. Distributed optimal controller synthesis for mixed mobility
4. Coordination of multiple autonomous vehicles in mixed traffic
5. Conclusion

The potential of autonomous vehicles on traffic dynamics

Mobility in 2019

- The emergence of autonomous vehicles is revolutionizing road transportation systems.

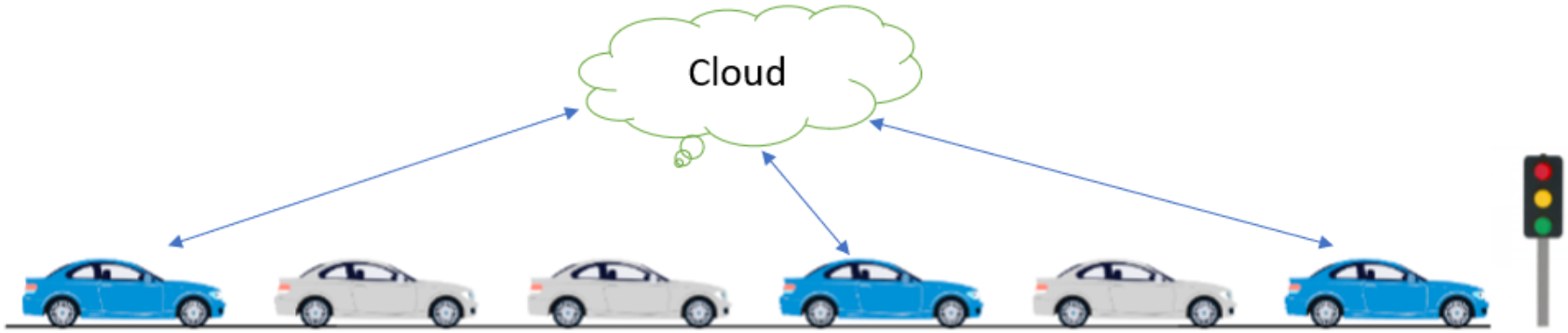
- Many big players are in the race of building fully autonomous vehicles. Volkswagen has been playing a leading role in the process.

Huge potential

- Reduce energy consumption, enhance traffic safety, and improve traffic efficiency
- New mobility patterns: ride-sharing, on-demand mobility, services to elderly and physically-challenged people.

Mixed Autonomy Mobility

Mixed autonomy mobility: a traffic condition where both autonomous vehicles and human-driven vehicles co-exist.



Main questions: can even a small scale of autonomous vehicles benefit traffic dynamics, and if so, how?

Traffic waves: theory & experiments

Main question: can even a small scale of autonomous vehicles benefit traffic dynamics, and if so, how?

- Starting from 1930s, there is a rich family of traffic flow theories to explain traffic jams, *e.g.*, partial differential equations, queuing theory, stochastic differential equations.

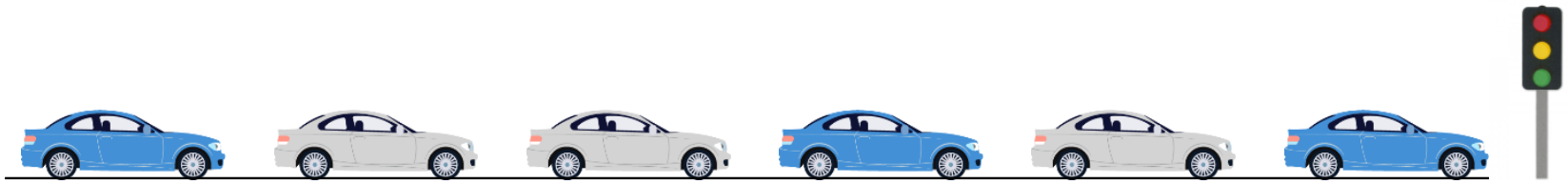
Experiments in 2008 $\Rightarrow$ **Experiments in 2018**

Sugiyama, Yuki, et al. 2008. Stern, Raphael E., et al. 2018.

State-of-the-art

Challenges in general mixed autonomy mobility

- *Complex dynamic models*: Cascaded, discontinuous, hybrid
- *Many interacting agents*: High-dimensional problems, information asymmetry
- *Human-behavior and reasoning*: Stochastic, self-interested, and non-cooperative

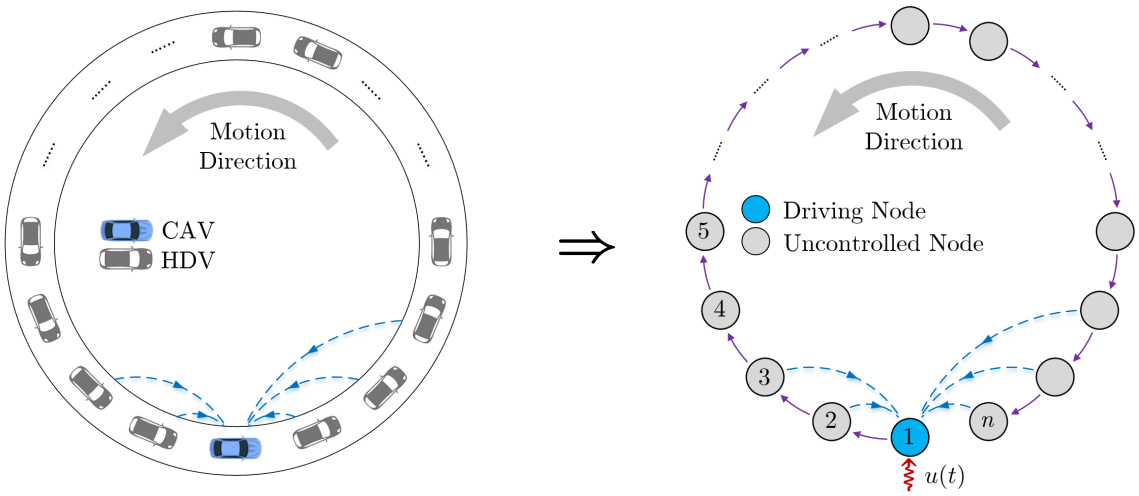


Selected literature

- **Simulation-based traffic flow analysis**: Bose, 2003; Shladover, 2012; Liu, 2018; Van Arem, 2006; Schakel, 2010; Van Driel, 2010; Wang, 2017; Calvert, 2017; Goñi-Ros, 2019
- **Theoretical analysis (mainly platooning level)**: A. Talebpour, H. Mahmassani, 2016; N. Mehr, R. Horowitz, 2018; **Cui et al, 2017**; G. Orosz, J. I. Ge et al. (Umich, from 2014)
- **Learning-based methods for mixed mobility**: A. M. Bayen's group at UCB.
  Wu, C., Kreidieh, A., Parvate, K., Vinitsky, E., & Bayen, A. M. (2017). Flow: Architecture and benchmarking for reinforcement learning in traffic control. arXiv preprint arXiv:1710.05465.

Part II: Control-theoretic modeling and analysis

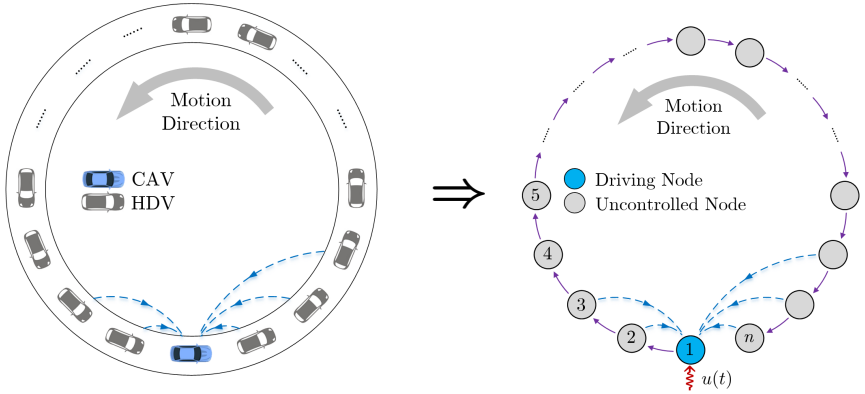
– Stability, controllability, and stabilizability



System modeling

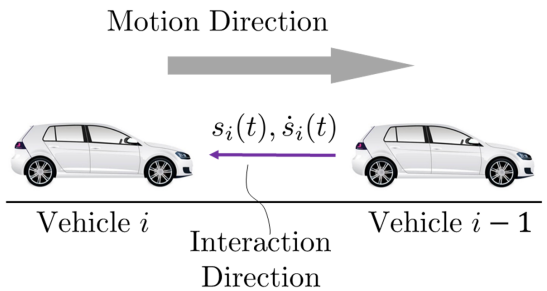
System setups

- a single-lane ring road with circumference L consisting of one AV and $n-1$ HDVs.
- The position, velocity and acceleration: p_i, v_i and a_i. The spacing of vehicle i: $s_i = p_{i-1} - p_i$.



HDV: car-following dynamics (e.g., OVM and IDM)

$$\dot{v}_i(t) = F_i\left(s_i(t), \dot{s}_i(t), v_i(t)\right), i = 2, \ldots, n,$$



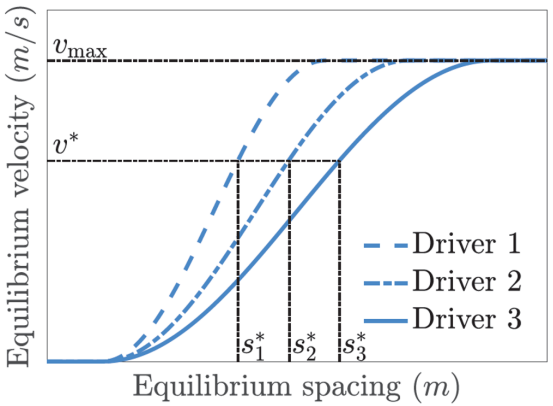
- Equilibrium points $\dot{s}_i(t) = 0, v_i = v^*, i = 2, \ldots, n$

$$F_i\left(s_i^*, 0, v^*\right) = 0$$

- Linearization: $\tilde{s}_i(t) = s_i(t) - s_i^*, \tilde{v}_i(t) = v_i(t) - v^*$.

$$\begin{cases} \dot{\tilde{s}}_i(t) = \tilde{v}_{i-1}(t) - \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = \alpha_{i1}\tilde{s}_i(t) - \alpha_{i2}\tilde{v}_i(t) + \alpha_{i3}\tilde{v}_{i-1}(t), \end{cases}$$

where $\alpha_{i1} = \frac{\partial F_i}{\partial s_i}, \alpha_{i2} = \frac{\partial F_i}{\partial \dot{s}_i} - \frac{\partial F_i}{\partial v_i}, \alpha_{i3} = \frac{\partial F_i}{\partial \dot{s}_i}$.



System modeling

Autonomous vehicle dynamics

- Control input $u_1(t) = a_1(t)$.
- Car-following dynamics: (s_1^* and v^* has no relationship!)

$$\begin{cases} \dot{\tilde{s}}_1(t) = \tilde{v}_n(t) - \tilde{v}_1(t), \\ \dot{\tilde{v}}_1(t) = u(t). \end{cases}$$

System dynamics

- Define the error state: $x(t) = \left[x_1^\mathsf{T}(t), x_2^\mathsf{T}(t), \ldots, x_n^\mathsf{T}(t)\right]^\mathsf{T}$
- Linearized model

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where

$$A = \begin{bmatrix} C_1 & 0 & \ldots & \ldots & 0 & C_2 \\ A_{22} & A_{21} & 0 & \ldots & \ldots & 0 \\ 0 & A_{32} & A_{31} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & A_{n2} & A_{n1} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ B_2 \\ \vdots \\ B_2 \end{bmatrix}$$

with each block matrix defined as

$$A_{i1} = \begin{bmatrix} 0 & -1 \\ \alpha_{i1} & -\alpha_{i2} \end{bmatrix}, A_{i2} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha_{i3} \end{bmatrix}, C_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Analysis I: Stability for all human-driven vehicles

Suppose all vehicles are human-driven and dynamics are homogeneous

$$\dot{x}(t) = \hat{A}x(t),$$

where

$$\hat{A} = \begin{bmatrix} A_1 & 0 & \dots & \dots & 0 & A_2 \\ A_2 & A_1 & 0 & \dots & \dots & 0 \\ 0 & A_2 & A_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_2 & A_1 & 0 \\ 0 & \dots & \dots & 0 & A_2 & A_1 \end{bmatrix}.$$

Define $\omega = e^{\frac{2\pi j}{n}}$, where j denotes the imaginary unit,

$$F_n^* = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

- Block-circulant matrix can be block-diagonalized

$$\hat{A} = (F_n^* \otimes I_2) \cdot \text{diag}(D_1, D_2, \dots, D_n) \cdot (F_n \otimes I_2)$$

For traffic systems with homogeneous HDVs only in a ring road, a stability condition is

$$\alpha_2^2 - \alpha_3^2 - 2\alpha_1 \geq 0.$$

This result first appeared in

- Cui, S., Seibold, B., Stern, R., & Work, D. B. (2017). Stabilizing traffic flow via a single autonomous vehicle: Possibilities and limitations. In Intelligent Vehicles Symposium (IV) (pp. 1336-1341). IEEE.

Analysis I: Stability for all human-driven vehicles

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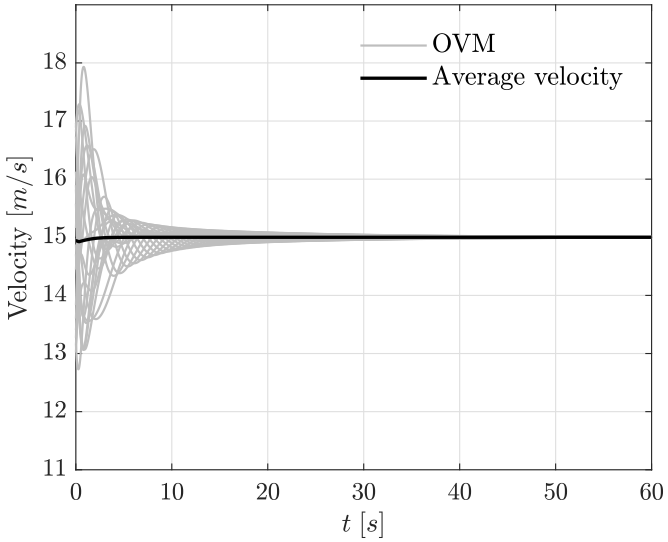
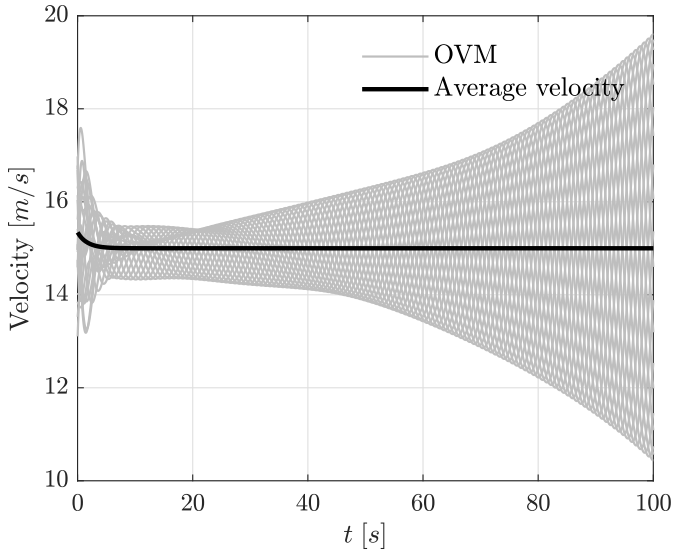
- For Optimal Velocity Model, $F(s_i(t), \dot{s}_i(t), v_i(t)) = \alpha(V(s_i(t)) - v_i(t)) + \beta\dot{s}_i(t)$, the stability condition is reduced to

$$\alpha + 2\beta \geq \dot{V}(s^*).$$



(a)



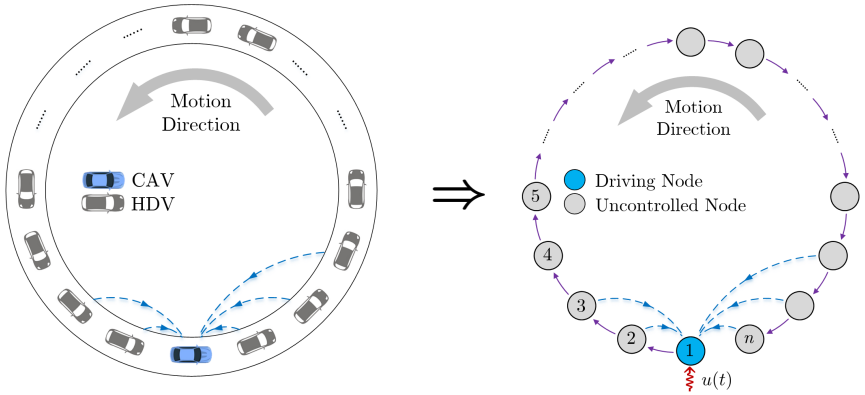
(b)

Analysis II: Controllability for mixed autonomy mobility

Linearized model

$$\dot{x}(t) = Ax(t) + Bu(t),$$

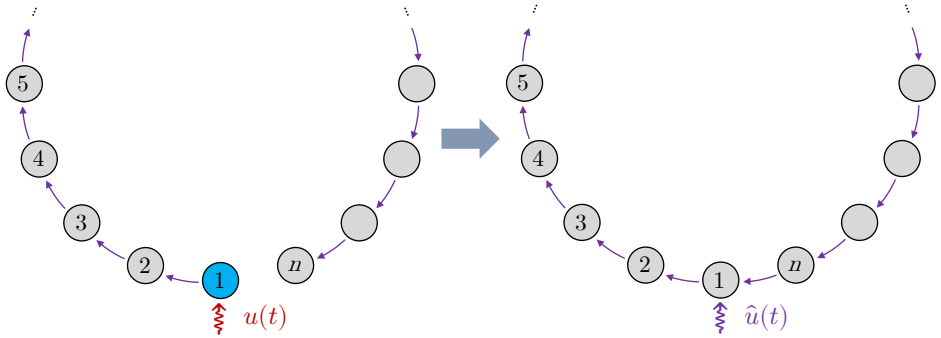
$$A = \begin{bmatrix} C_1 & 0 & \dots & \dots & 0 & C_2 \\ A_{22} & A_{21} & 0 & \dots & \dots & 0 \\ 0 & A_{32} & A_{31} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & A_{n2} & A_{n1} \end{bmatrix}$$



- **Controllability invariance** under state feedback and linear transformation

$$(A, B) \xrightarrow{\text{state feedback}} (\hat{A}, B) \xrightarrow{\text{linear transformation}} (\widetilde{A}, \widetilde{B})$$

1. Step 1: we apply $\hat{u}(t) = u(t) + Kx(t) = u(t) - (\alpha_1 \tilde{s}_1(t) - \alpha_2 \tilde{v}_1(t) + \alpha_3 \tilde{v}_n(t))$



Analysis II: Controllability for mixed autonomy mobility

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$$(A, B) \xrightarrow{\text{state feedback}} (\hat{A}, B) \xrightarrow{\text{linear transformation}} (\widetilde{A}, \widetilde{B})$$

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2. Step 2: Block-diagonalization using the Fourier matrix $\tilde{x} = (F_n \otimes I_2)x$

$$\dot{\tilde{x}} = \widetilde{A}\tilde{x}(t) + \widetilde{B}\hat{u}(t) = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix} \tilde{x}(t) + \frac{1}{\sqrt{n}} \begin{bmatrix} B_1 \\ B_1 \\ \vdots \\ B_1 \end{bmatrix} \hat{u}(t).$$

where

$$\frac{d}{dt}\begin{bmatrix} \tilde{x}_{i1} \\ \tilde{x}_{i2} \end{bmatrix} = \begin{bmatrix} 0 & -1 + \omega^{(n-1)(i-1)} \\ \alpha_1 & -\alpha_2 + \alpha_3 \omega^{(n-1)(i-1)} \end{bmatrix} \begin{bmatrix} \tilde{x}_{i1} \\ \tilde{x}_{i2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{n}} \end{bmatrix} \hat{u}(t), i = 1, \ldots, n.$$

- The first block is always uncontrollable! This corresponds to the ring structure (**zero eigenvalue**).

$$\tilde{x}_{11} = \frac{1}{\sqrt{n}} \left((s_1(t) - s_c^*) + \sum_{i=2}^{n} (s_i(t) - s^*) \right)$$

Analysis II: Controllability for mixed autonomy mobility

Theorem 1: Mixed traffic flow with homogeneous HDVs

- The mixed traffic flow in a ring road is not completely controllable. More precisely, we have

$$\text{rank}\left(\left[\widetilde{B}, \widetilde{A}\widetilde{B}, \ldots, \widetilde{A}^{2n-1}\widetilde{B}\right]\right) = \begin{cases} 2n-1, & \text{if } \alpha_1 - \alpha_2\alpha_3 + \alpha_3^2 \neq 0, \\ n, & \text{if } \alpha_1 - \alpha_2\alpha_3 + \alpha_3^2 = 0. \end{cases}$$

- The mixed traffic flow in a ring road is stabilizable!

Proof sketch:

- If $\alpha_1 - \alpha_2\alpha_3 + \alpha_3^2 \neq 0$, then there are no common eigenvalues between different diagonal blocks D_i and D_j. All the modes corresponding to non-zero eigenvalues are controllable using the PBH test ($\xi^\mathsf{T} A = \lambda A, \xi^\mathsf{T} B = 0, \xi \neq 0$)
- If $\alpha_1 - \alpha_2\alpha_3 + \alpha_3^2 \neq 0$, we have

$$\det(\lambda I - D_i) = (\lambda + \alpha_2 - \alpha_3)\left(\lambda + \alpha_3 - \alpha_3\omega^{(n-1)(i-1)}\right) = 0,\ i = 1, 2, \ldots, n,$$

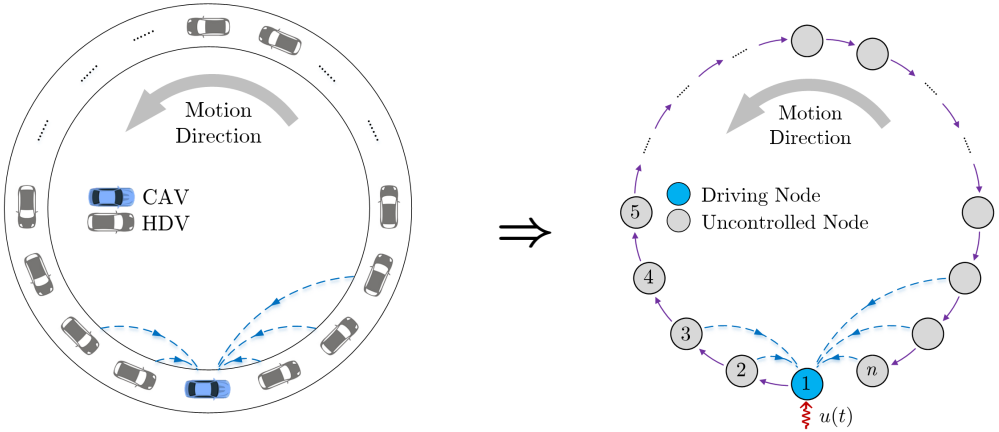
  The eigenvalue $\lambda = \alpha_3 - \alpha_2 < 0$ repeated n times and $n-1$ of them are uncontrollable but all stable.

Analysis II: Controllability for mixed autonomy mobility

Theorem 2: Mixed traffic flow with heterogeneous HDVs

- The mixed traffic flow in a ring road is not completely controllable.
- There exists one uncontrollable mode corresponding to a zero eigenvalue, and this uncontrollable mode is stable.
- It is stabilizable if we have,

$$\alpha_{j1}^2 - \alpha_{i2}\alpha_{j1}\alpha_{j3} + \alpha_{i1}\alpha_{j3}^2 \neq 0, \ \forall i, j \in \{1, 2, \ldots, n\}.$$



- In both cases, the uncontrollable mode corresponds to the ring structure

$$\rho_0^{\mathsf{T}} x(t) = \sum_{i=1}^{n} \tilde{s}_i(t) = \sum_{i=1}^{n} s_i(t) - \sum_{i=1}^{n} s_i^*.$$

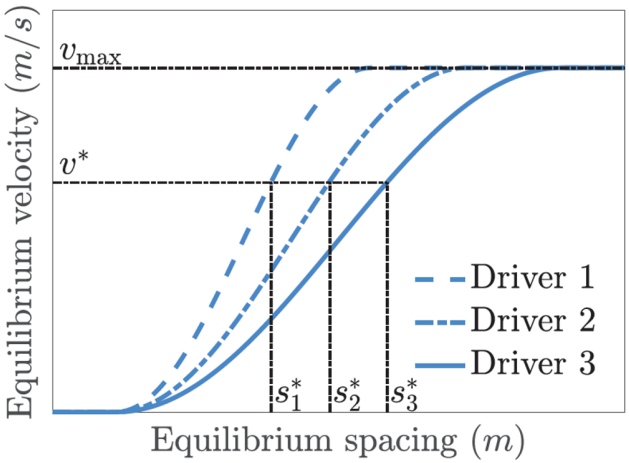
Analysis III: Reachability for mixed autonomy mobility

Question: How can a single autonomous vehicle guide the traffic flow to a desired (higher or lower) velocity v^*?

- Human-driven vehicles

$$F_i\left(s_i^*, 0, v^*\right) = 0 \Rightarrow v^* :\rightarrow s^*$$

- For autonomous vehicles, s_1^* can be designed seperately.



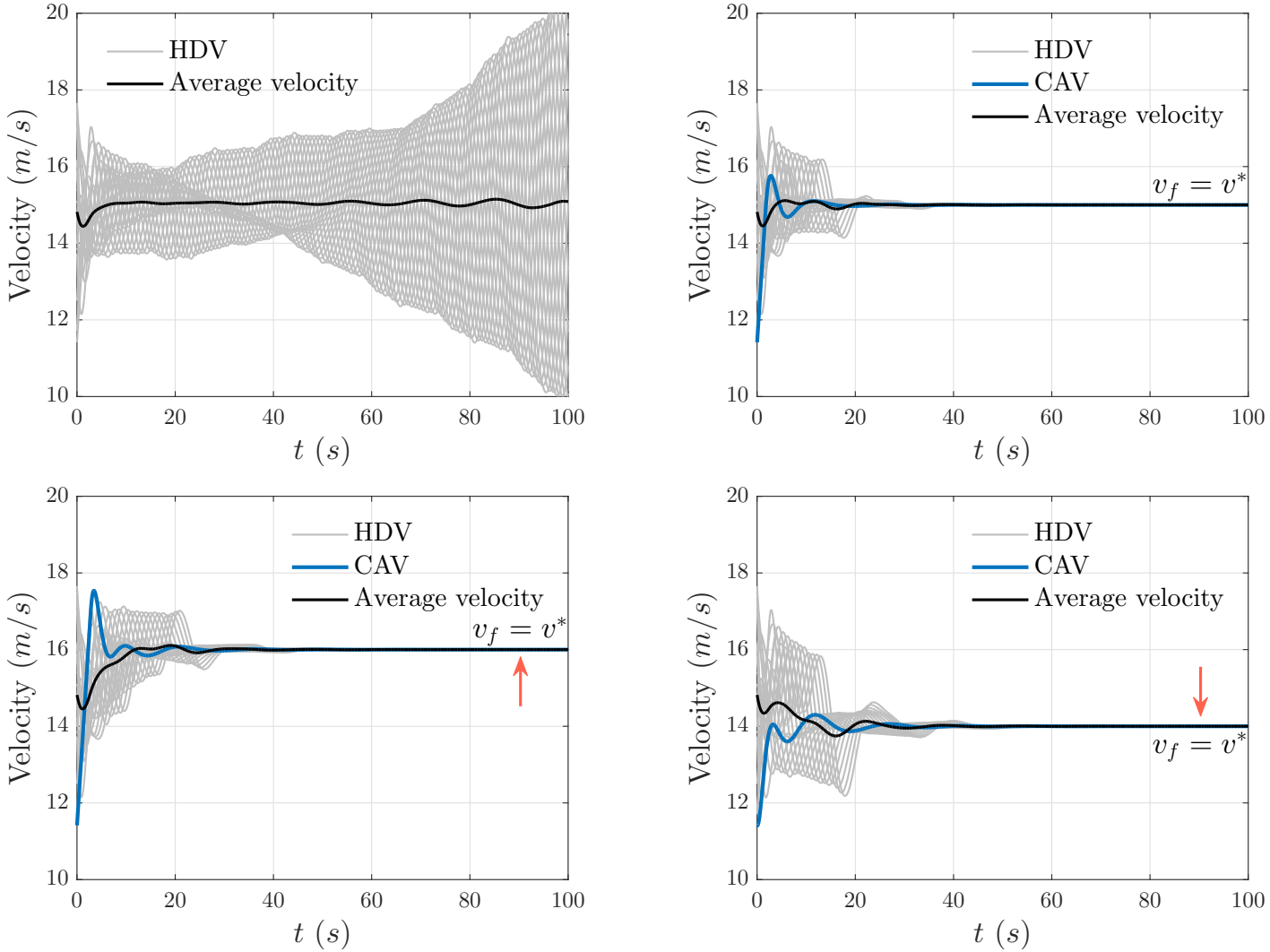
Theorem 3: Reachability for mixed autonomy mobility

- The traffic flow can reach the desired velocity v^* if and only if the desired spacing of the AV satisfies

$$s_1^* = L - \sum_{i=2}^{n} s_i^*.$$

- There is a reachable velocity range $0 \leq v^* < v_{\max}^*$. $\left(v_{\max}^* = F^{-1}\left(\frac{L}{n-1}\right)\right)$

Numerical results



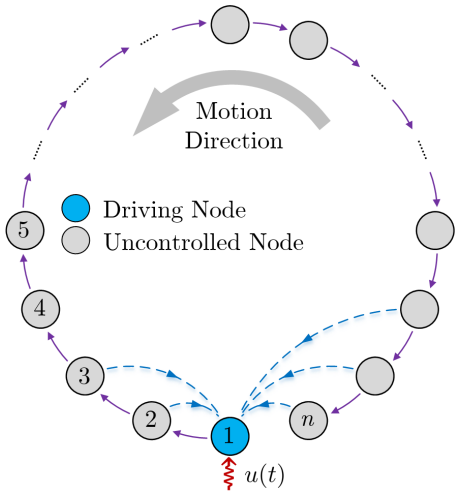
- The existence of **5% AVs** (1 out of 20) can bring **6%** improvement on traffic velocity

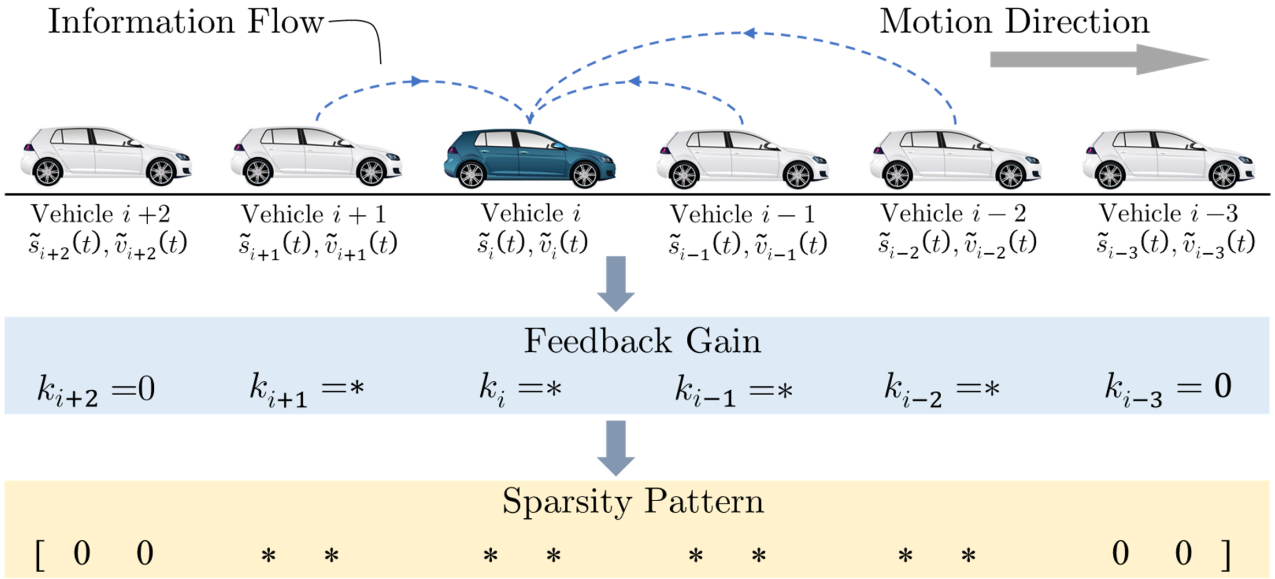
Part III: Distributed optimal controller synthesis for mixed mobility

Problem formulation: Distributed optimal control

Local available information

- due to limits of communication capabilities
- only partial information of the entire system is available
- Control law: $u(t) = -Kx(t)$, $K \in \mathcal{K}$ denotes a sparsity pattern





Problem formulation: Distributed optimal control

$\mathcal{H}_2$ optimal control

$$\dot{x}(t) = Ax(t) + Bu(t) + Hw(t)$$
$$z(t) = \begin{bmatrix} Q^{0.5} \\ -R^{0.5}K \end{bmatrix} x(t)$$

$$\begin{aligned} \min_{K} \quad & \|G_{z\omega}\|^2 \\ \text{subject to} \quad & u = -Kx, \\ & K \in \mathcal{K}, \end{aligned}$$

- Step 1: Standard reformulation

$$\begin{aligned} \min_{X,Y,Z} \quad & \mathrm{Tr}(QX) + \mathrm{Tr}(RY) \\ \text{subject to} \quad & AX + XA^\mathsf{T} - BZ - Z^\mathsf{T}B^\mathsf{T} + HH^\mathsf{T} \preceq 0, \\ & \begin{bmatrix} Y & Z \\ Z^\mathsf{T} & X \end{bmatrix} \succeq 0, \ X \succ 0, \ ZX^{-1} \in \mathcal{K}. \end{aligned}$$

- Step 2: sparsity invariance (Luca, Zheng, Papachristodoulou & Kamagarpour, 2019)

$$\begin{aligned} \min_{X,Y,Z} \quad & \mathrm{Tr}(QX) + \mathrm{Tr}(RY) \\ \text{subject to} \quad & AX + XA^\mathsf{T} - BZ - Z^\mathsf{T}B^\mathsf{T} + HH^\mathsf{T} \preceq 0, \\ & \begin{bmatrix} Y & Z \\ Z^\mathsf{T} & X \end{bmatrix} \succeq 0, \ X \succ 0, Z \in \mathcal{T}, \ X \in \mathcal{R}. \end{aligned}$$

Numerical experiments: damping traffic waves

- $n = 20, L = 400m$ one vehicle brakes at $3m/s^2$ for 3 seconds.
- Available information: 5 vehicles ahead and 5 vehicles behind

With all human-driven vehicles

With our controller

Numerical experiments: damping traffic waves

- $n = 20, L = 400m$ one vehicle brakes at $3m/s^2$ for 3 seconds.
- Available information: 5 vehicles ahead and 5 vehicles behind

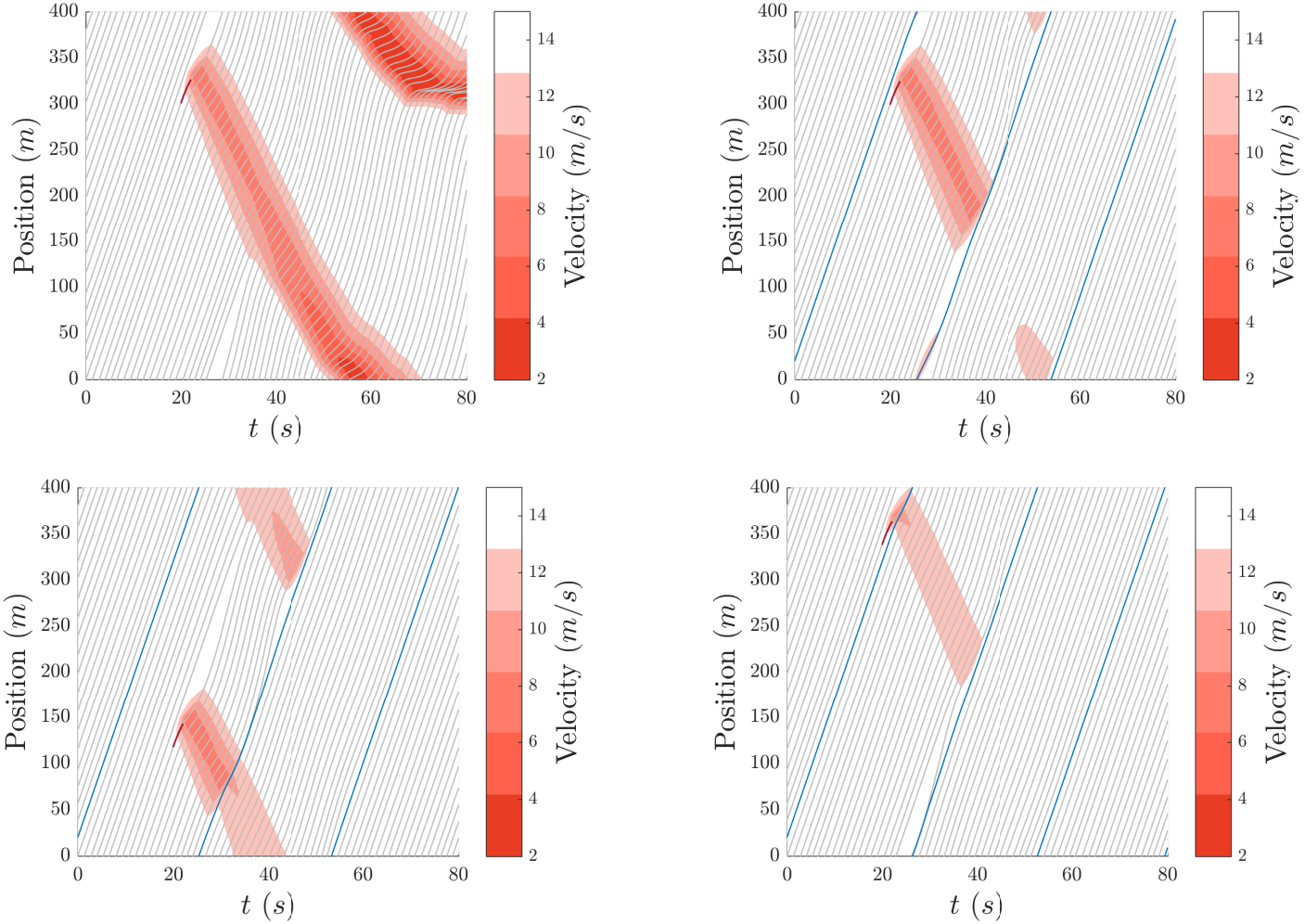
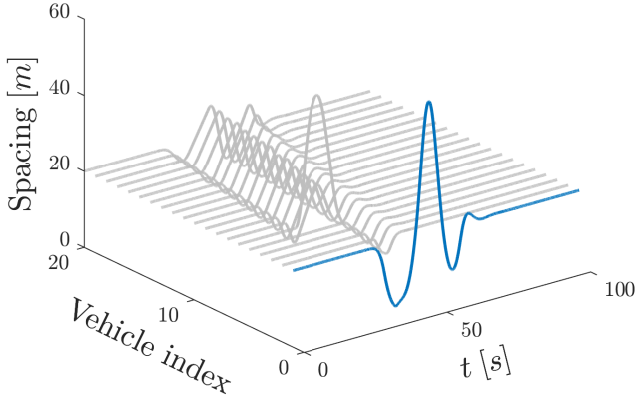


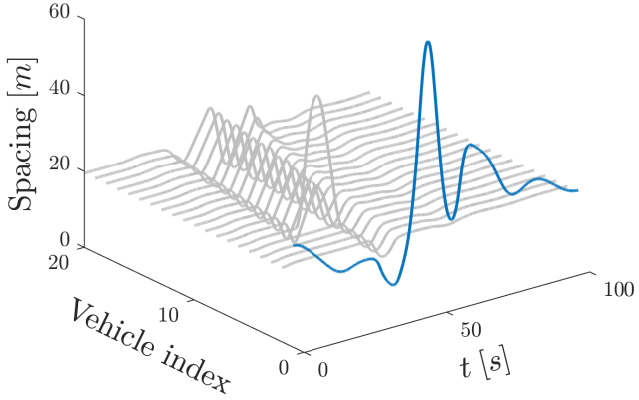
Figure: (a) All the vehicles are human-driven. (b)-(d) correspond to the cases where vehicle 2,11,20 is under the perturbation, respectively.

Numerical experiments: comparison

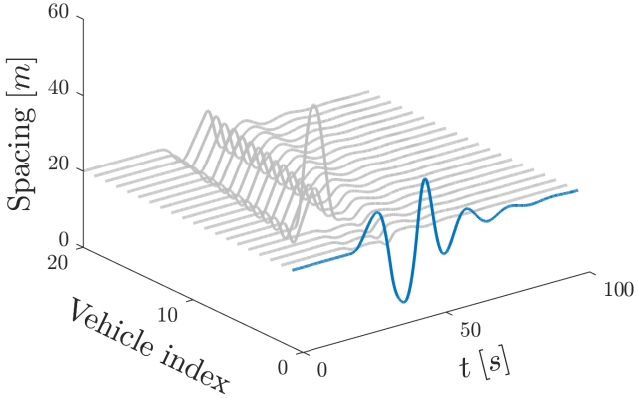
- Compare with empirical methods: FollowerStopper and PI with Saturation (Stern, 2018)



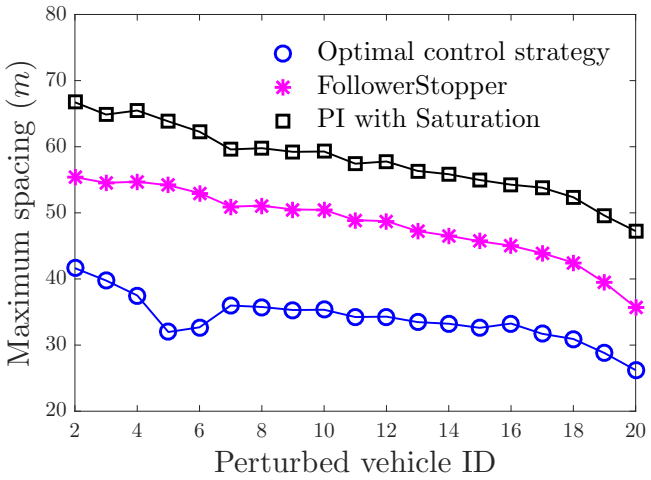
(a) FollowerStopper



(b) PI with Saturation

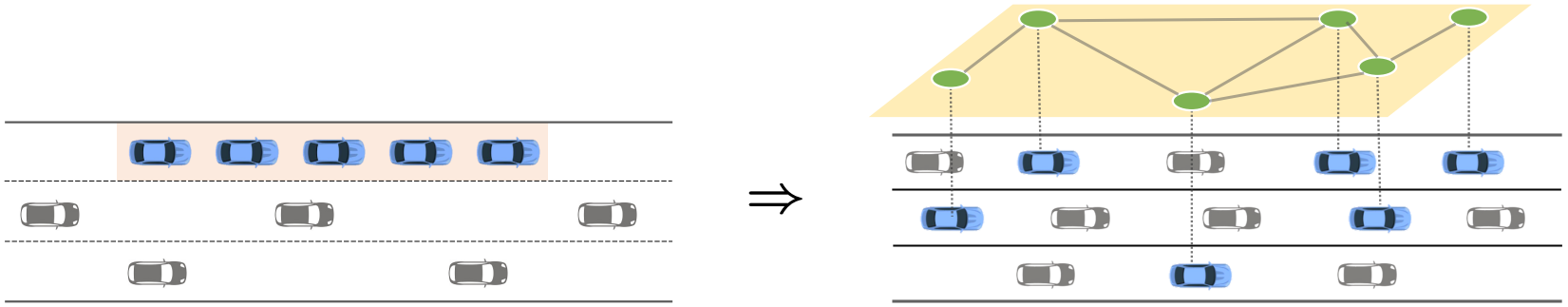


(c) Our method



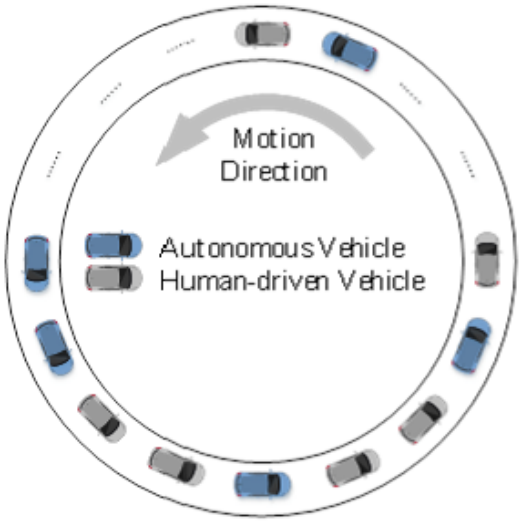
(d) Perturbation positions

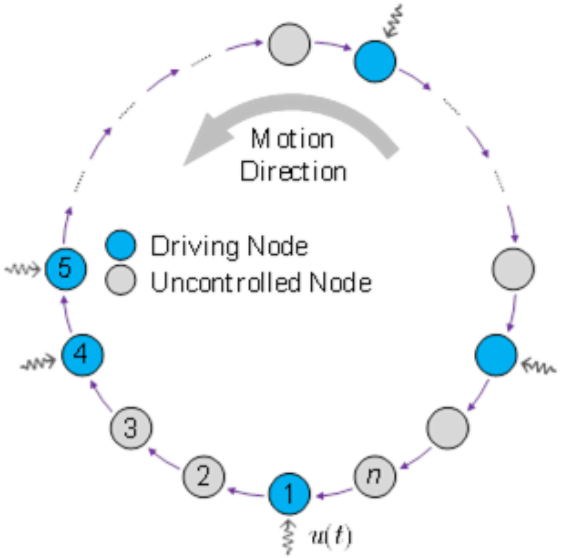
Part IV: Coordination of multiple autonomous vehicles in mixed traffic



Optimal coordination as set function optimization

Main question: How to coordinate multiple autonomous vehicles in traffic flow? Is platooning the optimal one?





Modeling

- Human-driven vehicles (linearization)

$$\begin{cases} \dot{\tilde{s}}_i(t) = \tilde{v}_{i-1}(t) - \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = \alpha_1 \tilde{s}_i(t) - \alpha_2 \tilde{v}_i(t) + \alpha_3 \tilde{v}_{i-1}(t), \end{cases}$$

- Autonomous vehicles

$$\begin{cases} \dot{\tilde{s}}_i(t) = \tilde{v}_i(t) - \tilde{v}_{i-1}(t), \\ \dot{\tilde{v}}_i(t) = u_i(t), \end{cases} \quad i \in S$$

Optimal coordination as set function optimization

Set function optimization

$$\max_{S} \quad J(S)$$
$$S \subseteq \Omega, |S| = k$$

where $\Omega = \{1, 2, \ldots, n\}$, S is the indices of autonomous vehicles, and $J : 2^{\Omega} \to \mathbb{R}$ denotes a utility function to be designed.

- **Submodularity:** (diminishing return property) a set function $f : 2^{\Omega} \to \mathbb{R}$ is called submodular if $\forall A \subseteq B \subseteq \Omega$ and $\forall e \in \Omega$, we have

$$f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B)$$

- **Monotonicity**: A set function $f : 2^{\Omega} \to \mathbb{R}$ is called non-increasing if for all $A \subseteq B \subseteq \Omega$, it holds that

$$f(A) \geq f(B)$$

- **Equivalent condition**: A set function $f : 2^{\Omega} \to \mathbb{R}$ is submodular if and only if the derived set functions $\Delta_f(\cdot \mid e) : 2^{\Omega \backslash \{e\}} \to \mathbb{R}$, defined as

$$\Delta_f(A \mid e) = f(A \cup \{e\}) - f(A), \quad \forall A \subseteq \Omega \backslash \{e\}$$

is non-increasing.

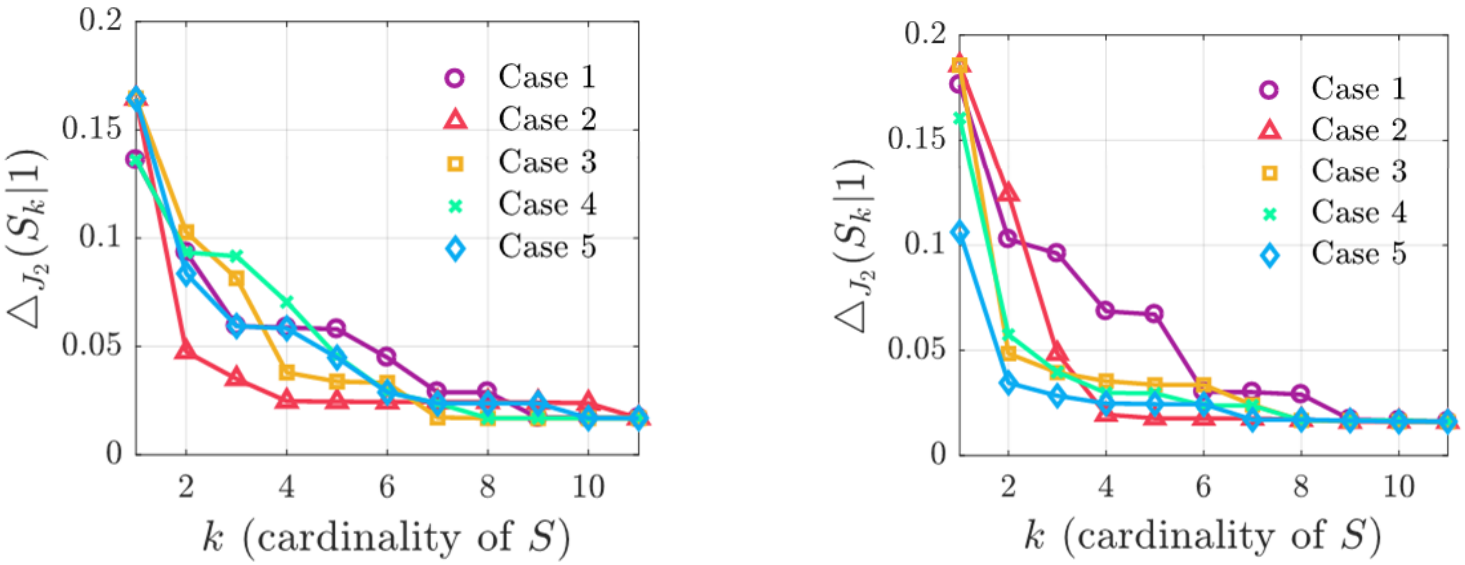
Optimal coordination as set function optimization

Case 1: each autonomous vehicle has a fixed CACC-type controller

- For each autonomous vehicle $i \in S$, we implement the controller

$$u_i(t) = (\alpha_1 - k_s)\tilde{s}_i(t) + (\alpha_2 - k_v)\tilde{v}_i(t) + \alpha_3 \tilde{v}_{i-1}(t)$$

- Closed-loop system $\dot{x}(t) = A_S x(t) + Hw(t), y(t) = Qx(t)$.
- Define the utility function as $J(S) = -\|G_{yw}\|_{\mathcal{H}_2}^2$



Penetration rate vs. Performance improvement: Introducing more autonomous vehicles with fixed controllers has diminishing improvements to traffic systems.

Optimal coordination as set function optimization

Case 2: redesign the controller for each autonomous vehicle

- Consider a state feedback controller $u(t) = -Kx(t)$.
- Closed-loop system becomes

$$\dot{x} = (A_S - B_S K)x + Hw$$

$$z = \begin{bmatrix} Q^{\frac{1}{2}} \\ -R^{\frac{1}{2}} K \end{bmatrix} x(t)$$

- Consider the utility function

$$J(S) := -\min_{K} \quad \|G_{zw}\|_{\mathcal{H}_2}^2$$

- Set function optimization for optimal coordination

$$\max_{S} \quad J(S)$$

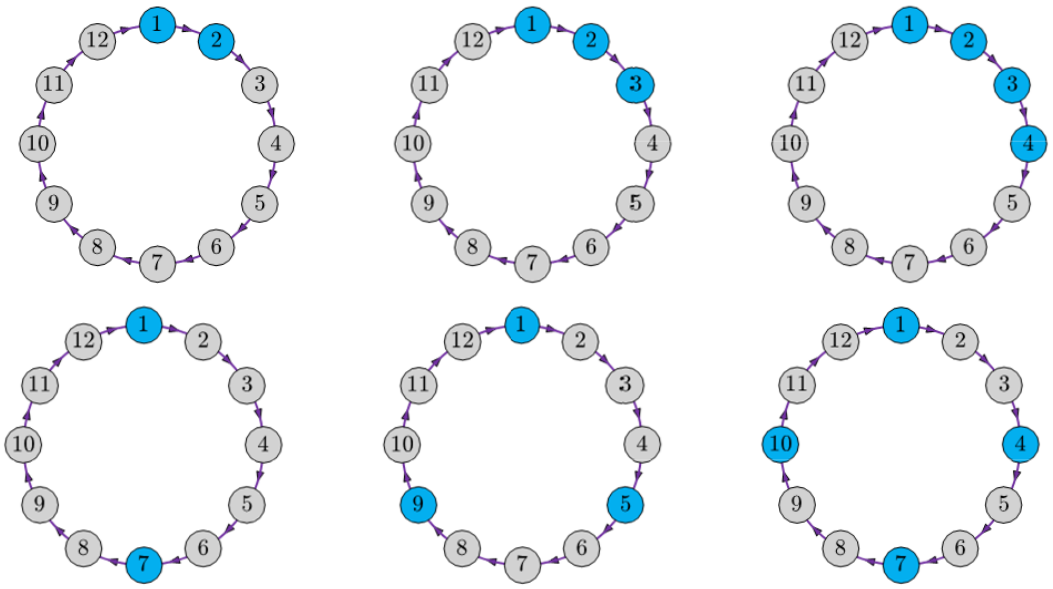
$$S \subseteq \Omega, |S| = k$$

- This set function is not submodular, as we find explicit counterexamples.
- In numerical simulations, we find two dominant patterns: **platooning** and **uniform distribution**.

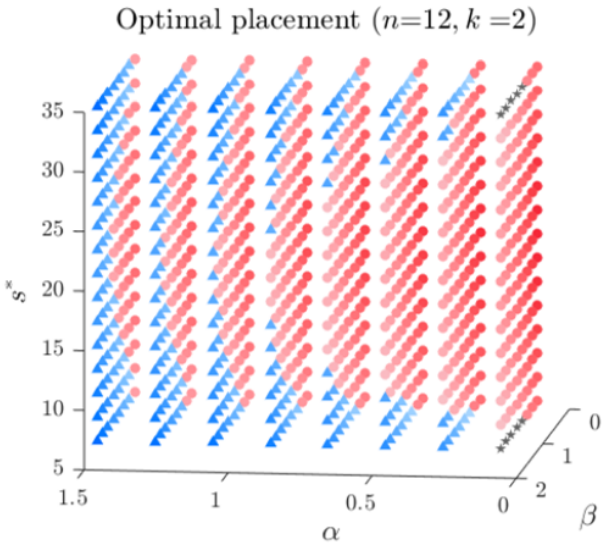
Two dominant patterns: platooning and uniform distribution

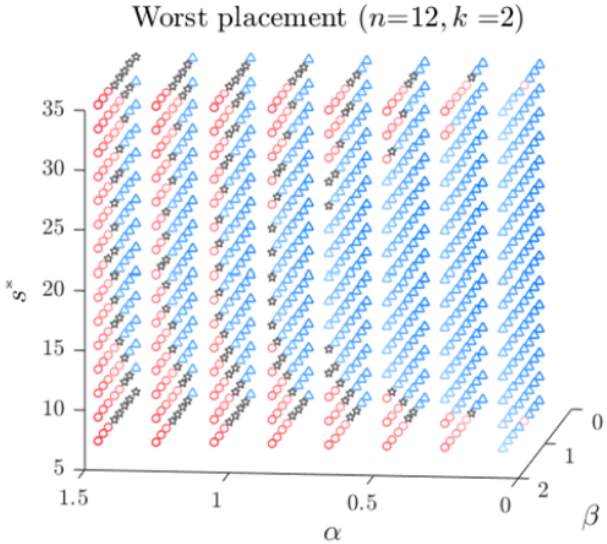
Platooning

Uniform distribution



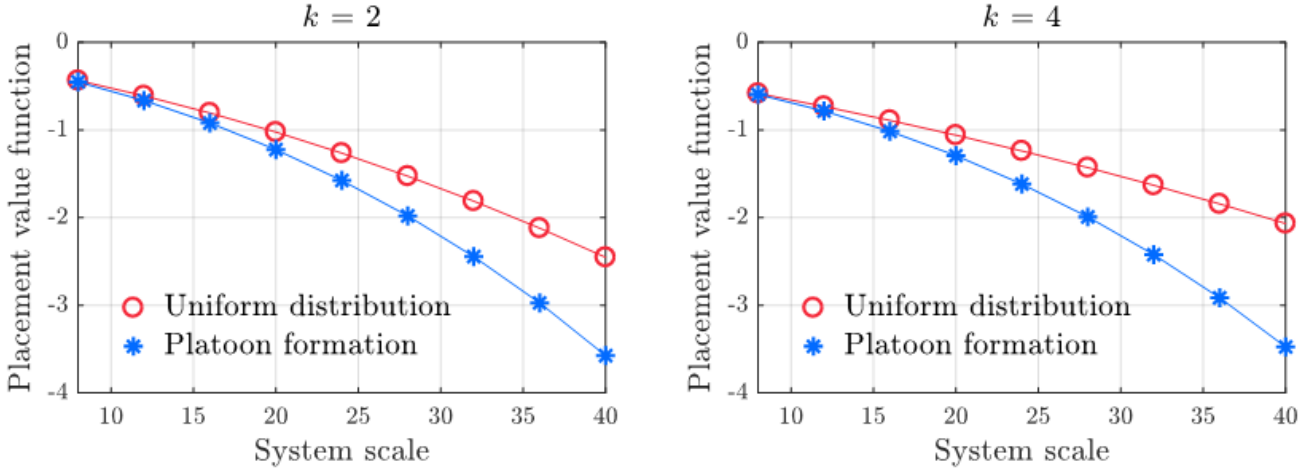
Red circles, blue triangles, and grey stars denote uniform distribution, platoon formation, and abnormal patterns, respectively



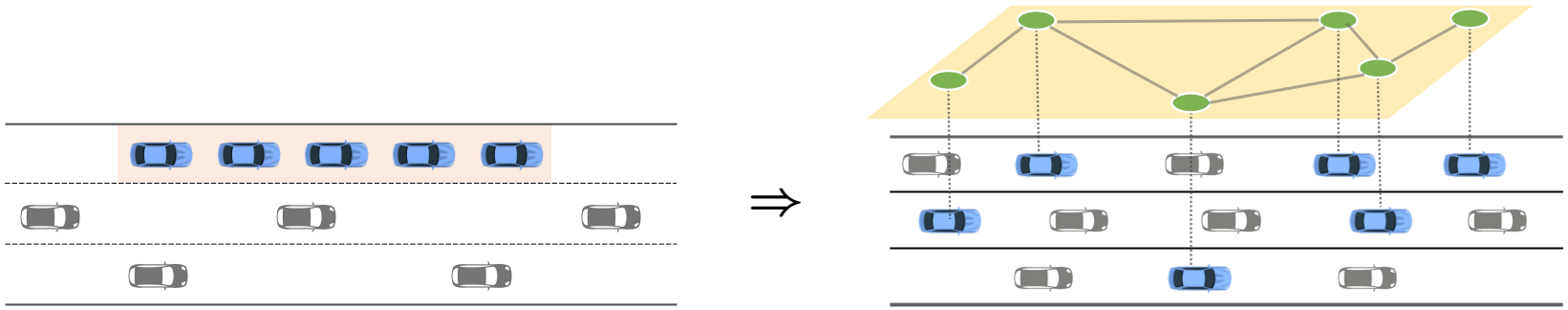


Coordination of autonomous vehicles beyond platooning

Performance at different scales



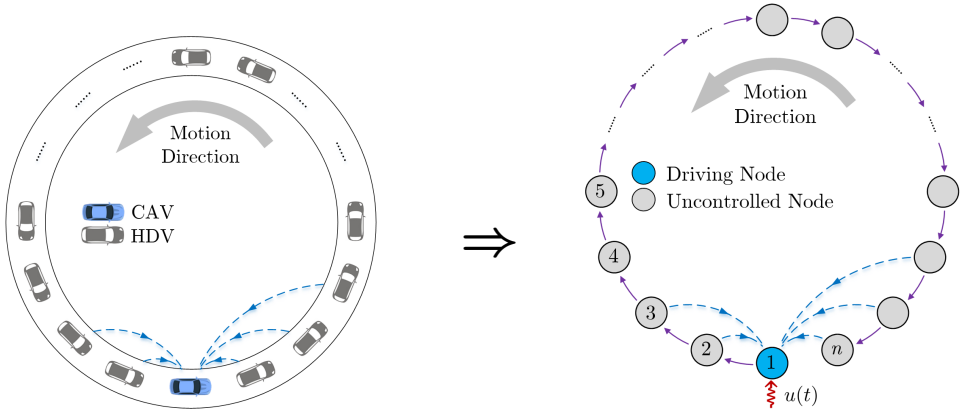
- The classical idea of platooning is not the optimal one across different scenarios, especially taking system-wide mobility into account.
- More opportunities beyond platooning.



Conclusion

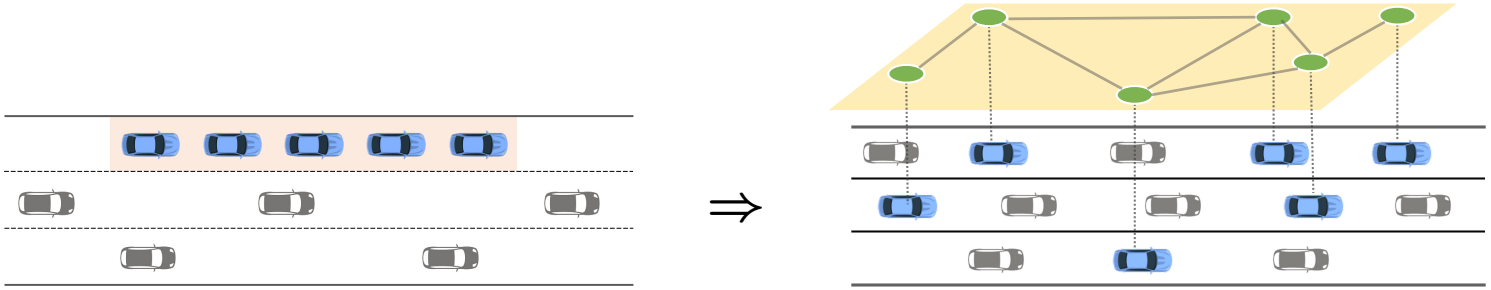
Take-home message

- **Message 1: control-theoretic analysis of mixed-autonomy mobility.** The following system is not completly cobtrollable, but is stabilizable.



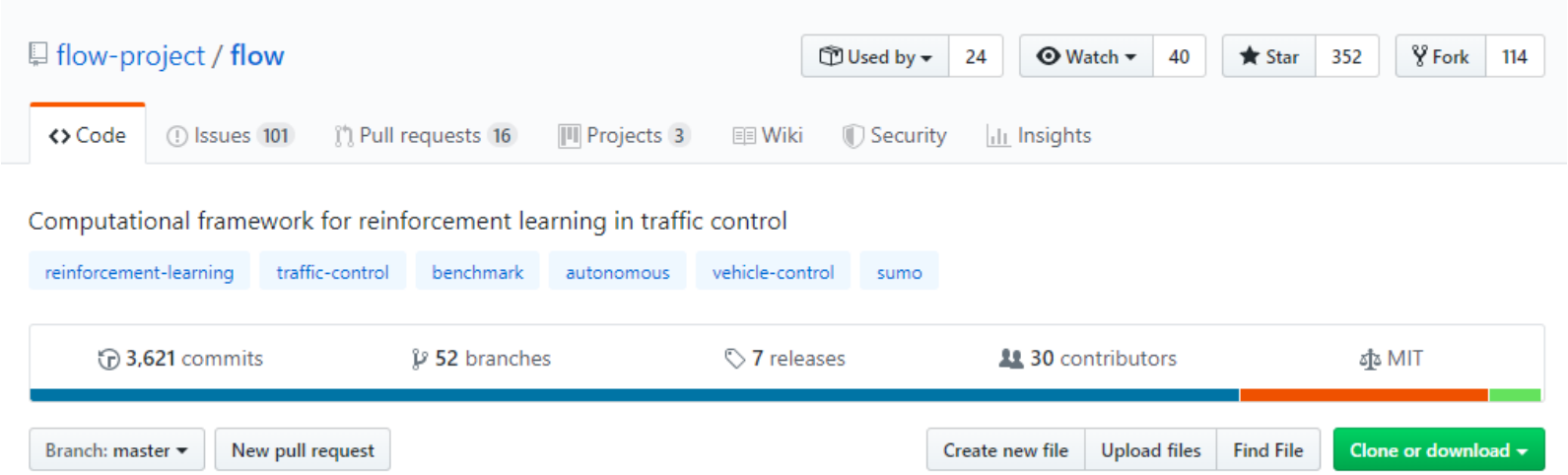
- **Message 2: The high potential of autonomous vehicles.** They can be used as mobile actuators to actively smooth and increase traffic velocity.

- **Message 3: Coordination of multiple autonomous vehicles beyond platooning.**



Topics beyond this talk

- **Unknown dynamics**: incorporate reinforcement learning in mixed autonomy.



- **Other scenarios**: multiple lanes, ramp metering, intersection, and other urban scenarios;
- **Rigorous safety guarantees**: e.g., develop distributed model predictive control for autonomous vehicles, where rigorous safety constraints can be encoded.

Thank you for your attention!

Q & A

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