Sample Complexity of Linear Quadratic Gaussian (LQG) Control for Output Feedback Systems

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Model-based LQG Formulation

Model: a *partially observed* LTI system

$$x_{t+1} = A_\star x_t + B_\star u_t + B_\star w_t, \quad y_t = C_\star x_t + v_t\,. \tag{1}$$

Optimal control problem:

$$\min_{u_0,u_1,\ldots} \lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T}\left(y_t^\mathsf{T} Q y_t + u_t^\mathsf{T} R u_t\right)\right] \quad \text{subject to} \quad (1)\,. \tag{2}$$

Goal: Find optimal linear control policy

$$\mathbf{u}(z) = \mathbf{K}(z)\mathbf{y}(z)\,,$$

minimizing (2).

Solution: Classical, e.g. observer/controller design with (model-based) Riccati equations.

What if the model is completely unknown?

LQG - Unknown Model

Standard Assumption: Stable plant, *i.e.*,

$$\rho(A_\star) < 1\,.$$

Proposed Design Procedure

I) Compute an estimate $\hat{\mathbf{G}}$ of the input/output *impulse response*

$$\mathbf{G}_\star(z) = C_\star(zI - A_\star)^{-1}B_\star\,.$$

Let $\epsilon = \|\mathbf{\Delta}\|_\infty := \|\mathbf{G}_\star - \hat{\mathbf{G}}\|_\infty$ be the estimation error.

II) Design a near-optimal controller...

- robustly stabilizing for all $\|\mathbf{\Delta}\|_\infty \le \epsilon$

Main contributions

1. **End-to-end sample complexity bound on learning a robust LQG controller** for open-loop stable plants.
2. **Convex design of robust output-feedback controllers**, enabled by the Input-Output Parametrization (IOP) [1].
3. **LQG performance degrades linearly with the estimation error.**

Alternative Design Philosophies

- **Idea 1**: First estimate $\hat{A}, \hat{B}, \hat{C}$, then design optimal LQG controller (*Certainty Equivalent Controller* [2])
  → **may not stabilize $(A^\star, B^\star, C^\star)$!**
- **Idea 2**: So... First estimate
  $$\|\hat{A} - A_\star\|, \quad \|\hat{B} - B_\star\|, \quad \|\hat{C} - C_\star\|,$$
  and then design a *robustly stabilizing* controller...

This is non-trivial!

- SLS from [3] can deal with *state-feedback*
- State-space estimation errors are only valid up to an **unknown** change of variables [2]

I) Nonasymptotic Identification

- **Estimate Markov parameters**
  $$\hat{\mathbf{G}} = \begin{bmatrix}\hat{C}\hat{B} & \hat{C}\hat{A}\hat{B} & \cdots \hat{C}\hat{A}^{T-1}\hat{B}\end{bmatrix}$$
  using standard least-squares identification
- **Adapt nonasymptotic bound from [4]**
  $$\|\mathbf{G}_\star - \hat{\mathbf{G}}\|_\infty \le \frac{R_w + R_v + R_e}{\sigma_u}\sqrt{\frac{T}{N}} + \Phi(A_\star)\,\|C_\star\|\,\|B_\star\|\,\frac{\rho(A_\star)^T}{1-\rho(A_\star)}.$$

→ $\sqrt{\frac{T}{N}}$: error 1) *increases* with modelling complexity T and 2) *decreases* with more data N

II) Robust Controller Synthesis

1. Input-Output Parametrization (IOP) [1]

Optimize over $\mathbf{K}$
≡
Optimize over *closed-loop responses* $(\mathbf{Y}, \mathbf{U}, \mathbf{W}, \mathbf{Z})$

$$\begin{bmatrix}\mathbf{Y} & \mathbf{W}\\ \mathbf{U} & \mathbf{Z}\end{bmatrix} := \begin{bmatrix}(I - \mathbf{G}_\star\mathbf{K})^{-1} & (I - \mathbf{G}_\star\mathbf{K})^{-1}\mathbf{G}_\star\\ \mathbf{K}(I - \mathbf{G}_\star\mathbf{K})^{-1} & (I - \mathbf{K}\mathbf{G}_\star)^{-1}\end{bmatrix}.$$

2. Tractable Robust Optimization (▲)

$$\min_{\gamma\in[0,1/\epsilon)} \frac{1}{1-\epsilon\gamma} \min_{\hat{\mathbf{Y}},\hat{\mathbf{W}},\hat{\mathbf{U}},\hat{\mathbf{Z}}} \left\|\begin{bmatrix}\sqrt{1+h(\epsilon,\alpha)}\hat{\mathbf{Y}} & \hat{\mathbf{W}}\\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix}\right\|_{\mathcal{H}_2}$$

subject to

$$\begin{bmatrix}I & -\hat{\mathbf{G}}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{Y}} & \hat{\mathbf{W}}\\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix} = \begin{bmatrix}I & 0\end{bmatrix},$$

$$\begin{bmatrix}\hat{\mathbf{Y}} & \hat{\mathbf{W}}\\ \hat{\mathbf{U}} & \hat{\mathbf{Z}}\end{bmatrix}\begin{bmatrix}-\hat{\mathbf{G}}\\ I\end{bmatrix} = \begin{bmatrix}0\\ I\end{bmatrix} \quad \hat{\mathbf{Y}}, \hat{\mathbf{W}}, \hat{\mathbf{Z}} \in \mathcal{RH}_\infty,$$

$$\|\hat{\mathbf{U}}\|_\infty \le \min(\gamma, \alpha)\,.$$

Remarks

- Program (▲) is *quasi-convex*
  - Inner program is convex
  - Golden-ratio search over γ
- Constraint on $\left\|\hat{\mathbf{U}}\right\|_\infty$ yields *robustness* [5]
  - Controller $\hat{\mathbf{K}} = \hat{\mathbf{U}}\hat{\mathbf{Y}}^{-1}$ stabilizes all $\hat{\mathbf{G}}$ with $\left\|\hat{\mathbf{G}} - \mathbf{G}_\star\right\|_\infty \le \epsilon^{-1}$
- α is an hyper-parameter to be tuned

Main Theorem - Sample Complexity

If ϵ is small enough, the controller $\mathbf{K} = \hat{\mathbf{U}}_\star\hat{\mathbf{Y}}_\star^{-1}$, which is optimal for (▲), is such that

i) $\mathbf{K}$ stabilizes the true plant $\mathbf{G}_\star$,

ii) $\mathbf{K}$ introduces a suboptimality of at most

$$\frac{J(\mathbf{G}_\star, \mathbf{K})^2 - J(\mathbf{G}_\star, \mathbf{K}_\star)^2}{J(\mathbf{G}_\star, \mathbf{K}_\star)^2} \le 20\epsilon\|\mathbf{U}_\star\|_\infty + \mathcal{O}(\epsilon)\,,$$

or, in terms of how much data is available:

$$\frac{J(\mathbf{G}_\star, \mathbf{K})^2 - J(\mathbf{G}_\star, \mathbf{K}_\star)^2}{J(\mathbf{G}_\star, \mathbf{K}_\star)^2} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

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